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Non-Abelian BF theory for 2 + 1 dimensional topological states of matter

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\textbf{Abstract.} We present a field theoretical analysis of the 2 + 1 dimensional BF model with boundary in the Abelian and the non-Abelian case based on Symanzik’s separability condition. Our aim is to characterize the low-energy properties of time reversal invariant topological insulators. In both cases, on the edges, we obtain Kač–Moody algebras with opposite chiralities reflecting the time reversal invariance of the theory. While the Abelian case presents an apparent arbitrariness in the value of the central charge, the physics on the boundary of the non-Abelian theory is completely determined by time reversal and gauge symmetry. The discussion of the non-Abelian BF model shows that time reversal symmetry on the boundary implies the existence of counter-propagating chiral currents.

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1. Introduction

During the last few decades, a new paradigm, based only on the topological properties of a system, has been investigated to classify and predict new states of matter. The starting point was, in the early 1980s, the discovery of the quantum Hall effect (QHE), a phase peculiar to 2 + 1 dimensions (D) [1], where first appeared the connection between a macroscopical physical quantity, i.e. the Hall conductivity, and a class of topological invariants [2]. This was the first example of new kinds of materials which cannot be described in terms of the principle of broken symmetries. Recently, a new class of systems was discovered that, analogous to the QHE, behave like insulators in the bulk, but support robust conducting edge or surface states, hence displaying non-trivial topological properties. The low-energy sector of this kind of state is well described in terms of topological field theories (TFT) [3]. The QHE was successfully described by the Abelian Chern Simons (CS) theory both in the integer and in the fractional regime [4, 5]. Moreover, the non-Abelian CS theory has been proposed for more exotic fractional states [5] that are predicted to have excitations with non-Abelian statistics, a key point for the topologically protected quantum computation [6]. Restriction to the boundary of these TFT has been discussed in order to describe the dynamics of the edge states of the system [5]. In contrast with the QHE, where the magnetic field breaks time reversal (T) symmetry, a new class of T invariant systems, i.e. topological insulators (TI), has been predicted [7, 8] and experimentally observed [9] in 2 + 1 D, leading to the quantum spin Hall (QSH) effect. At the boundary of these systems, one has helical states, namely electrons with opposite spin propagating in opposite directions. The presence of these edge currents with opposite chiralities leads to an extremely peculiar current–voltage relationship in multi-terminal measurements. The experimental observations carried out by the Molenkamp group [9, 10] support these theoretical predictions. Also non-trivial realization of TI in 3 + 1 D has been conjectured and realized [11, 12]. States with T and parity (P) symmetry emerge naturally also in lattice models of correlated electrons [13, 14] where doubled CS field theories were developed. In the case of 2 + 1 D TI an Abelian doubled CS [15, 16] or, equivalently, a BF effective theory has been introduced [17, 18]. The latter TFT allows generalizations to higher dimensions and represents a promising candidate for the description of 3 + 1 D TI [17]. Despite the great potential of the BF model in describing T invariant topological states of matter, a careful discussion of these theories in the presence of a boundary is still lacking.
In this paper, we present a detailed and systematic derivation of the helical states at the boundary of the 2 + 1 D Abelian and non-Abelian BF model, applying Symanzik’s method [19]. This approach represents the most natural way to introduce a boundary in a quantum field theory and was already considered with success in a different context for the CS case [20–22]. It is known that 2 + 1 D CS theory on a manifold with boundary displays a conformal structure [23]. This is a general property of TFTs. The BF theory, once a T invariant boundary is introduced, displays two Kač–Moody algebras with opposite chiralities, according to the physics of the QSH effect [8, 9], both in the Abelian and in the non-Abelian case [24].

2. The Abelian BF model

Here, we recall the main properties of the 2 + 1 D Abelian BF model. Its action depends on two Abelian gauge fields \( A \) and \( B \) and, in the Minkowski spacetime, can be written as

\[
S_{bf} = \frac{k}{2\pi} \int d^3 x \epsilon^{\mu\nu\rho} F_{\mu\nu} B_{\rho}
\]

with \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( k \) an integer due to the invariance of the partition function under large gauge transformations, in exactly the same way as happens in the CS case [3]. At low energy, the action (1) is the only renormalizable one, involving two gauge fields, which is invariant under gauge transformations in 2 + 1 dimensions. At this point the T invariance appears to be a discrete symmetry of the theory together with the trivial exchange \( A_\mu \leftrightarrow B_\mu \). In order to describe the model in the presence of a planar boundary it is useful to rewrite (1) in the so-called light-cone coordinates \( u = y, z = (t + x)/\sqrt{2} \) and \( \bar{z} = (t - x)/\sqrt{2} \) where the field components become \( (A_u, A, A_u), (B_u, B, B_u) \). In these new variables the action reads

\[
S_{bf} = \frac{k}{\pi} \int du d^2 z [B (\bar{\partial} A_u - \partial u \bar{A}) + \bar{B} (\partial A_u - \partial u A) + B_u (\bar{\partial} A - \bar{\partial} A)].
\]

The mentioned T symmetry (modulo an exchange of the two gauge fields) in these new variables is

\[
(u, z, A, A_u, B, B_u) \leftrightarrow (u, -\bar{z}, -A, A_u, \bar{B}, -B_u);
\]

note that the \( A \) and \( B \) fields transform differently under T, which signals the fact that they can be linked to different physical quantities. Indeed, this allows us to interpret \( A \) as a charge density and \( B \) as a spin density, the fundamental ingredients of the description of TI [17].

As is well known, gauge field theories are affected by the presence of redundant degrees of freedom, and a gauge fixing choice is necessary in order to deal with the physical degrees of freedom only. The gauge fixing procedure introduces in the theory unphysical ‘ghost fields’. Now, while in the Abelian case the ghost fields decouple, and can be integrated out from the Green functions generating functionals, this does not happen in the non-Abelian case, where ghost fields are truly quantum, although unphysical, fields [25]. After gauge fixing, the propagators of the theory can be defined and the quantum extension of the partition function can be discussed.

Covariance being already broken by the presence of a boundary, it is convenient to adopt the axial gauge choice \( A_u = B_u = 0 \), and add to (1) the corresponding gauge fixing term \( S_{gf} \). The coupling with external sources is introduced by means of the term \( S_{ext} = \int du d^2 z \sum_\psi j_\psi \psi \psi \), with \( \psi \) being a generic field of the theory.
3. The presence of a boundary

We consider now the planar boundary \( u = 0 \). The introduction of a boundary in field theory was thoroughly discussed a long time ago by Symanzik [19]. The basic idea is quite simple and general, and concerns the propagators of the theory, which only in the gauge fixed theory are well defined: the propagators of the theory between points lying on opposite sides of the boundary must vanish. This *separability* condition results in the following general form of the propagator for any field \( \psi \):

\[
\Delta_{\psi, \psi'} = \Theta(u) \Theta(u') \Delta_{\psi, \psi'} + \Theta(-u) \Theta(-u') \Delta_{\psi, \psi'},
\]

where \( \pm \) indicates the value of the quantities for \( (u \to 0^\pm) \) and \( \Theta(u) \) is the Heaviside step function. In addition, the usual field theory constraints of locality and power counting are required together with helicity conservation [20].

The most general boundary action consistent with the previous conditions is

\[
S_{bd} = -\frac{k}{\pi} \int du \, d^2z \, \delta(u)(\alpha_1 A \bar{A} + \alpha_2 A \bar{B} + \alpha_3 \bar{A} B + \alpha_4 B \bar{B}),
\]

where \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \) are free real parameters. The above boundary term leads to \( \delta(u) \)-dependent breakings of the equations of motion

\[
\mathcal{F}_{\bar{B}} = \sum_{\eta = \pm} \frac{k}{\pi} \delta(u)(\alpha_2 A_\eta + \alpha_4 B_\eta),
\]

\[
\mathcal{F}_{\bar{B}} = \sum_{\eta = \pm} \frac{k}{\pi} \delta(u)(\alpha_3 \bar{A}_\eta + \alpha_4 \bar{B}_\eta),
\]

\[
\mathcal{F}_{\bar{B}} = \sum_{\eta = \pm} \frac{k}{\pi} \delta(u)(\alpha_1 A_\eta + \alpha_3 B_\eta),
\]

\[
\mathcal{F}_{\bar{B}} = \sum_{\eta = \pm} \frac{k}{\pi} \delta(u)(\alpha_1 \bar{A}_\eta + \alpha_2 \bar{B}_\eta).
\]

where \( \mathcal{F}_{\bar{B}} = \delta S / \delta \psi \) and \( S = S_{bf} + S_{gf} + S_{ext} \).

From now on, we will focus on the ‘+’ side of the boundary, the ‘−’ side being obtained by the P symmetry \( (u, z, A, \bar{A}, B, \bar{B}) \leftrightarrow (-u, \bar{z}, \bar{A}, A, \bar{B}, B) \).

In order to fix the parameters, we integrate (6)–(9) at vanishing sources with respect to \( u \) in the infinitesimal interval \( (-\epsilon, \epsilon) \). This leads to

\[
(1 - \alpha_2) A_+ - \alpha_4 B_+ = 0,
\]

\[
\alpha_1 A_+ - (1 - \alpha_3) B_+ = 0,
\]

\[
(1 + \alpha_3) \bar{A}_+ + \alpha_4 \bar{B}_+ = 0,
\]

\[
\alpha_1 \bar{A}_+ + (1 + \alpha_2) \bar{B}_+ = 0.
\]

It is easy to verify that (10)–(13) have non-trivial T invariant solutions when the determinants vanish, namely

\[
\alpha_1 \alpha_4 - (1 - \alpha_2^2) = 0; \quad \alpha_2 = -\alpha_3.
\]
The Abelian BF theory with boundary satisfies the following Ward identities (WI) on the generating functional $Z_c$:

$$\frac{\pi}{k} \int du \, H[Z_c] \equiv -\frac{\pi}{k} \int du (\tilde{\partial} j_A + \partial j_A)$$

$$= -[\alpha_1 (\tilde{\partial} A_\alpha + \partial \tilde{A}_\alpha) + \alpha_2 (\tilde{\partial} B_\alpha - \partial \tilde{B}_\alpha)],$$

(15)

$$\frac{\pi}{k} \int du \, N[Z_c] \equiv -\frac{\pi}{k} \int du (\tilde{\partial} j_B + \partial j_B)$$

$$= -[\alpha_2 (\tilde{\partial} A_\alpha - \partial \tilde{A}_\alpha) + \alpha_4 (\tilde{\partial} B_\alpha + \partial \tilde{B}_\alpha)].$$

(16)

It is well known that the axial gauge is not a complete gauge fixing [26]: a residual gauge invariance remains, expressed by the two WI (15)–(16), one for each gauge field $A$ and $B$. As is apparent, the presence of the boundary results in $\delta(u)$-dependent linear breaking terms on the rhs of (15)–(16). Such linear terms are allowed, since in quantum field theory a non-renormalization theorem ensures that linear breakings are present at the classical level only, and do not acquire quantum corrections [27].

In other words, the WI (15)–(16) represent the most general expression of the gauge invariance of the complete theory (bulk and boundary).

4. Conserved currents and algebra

We introduce the fields

$$R_+ \equiv (1 - \alpha_2) A_\alpha + \alpha_4 B_\alpha,$$

(17)

$$S_+ \equiv (\alpha_2 - 1) A_\alpha + \alpha_4 B_\alpha,$$

(18)

which, according to (10)–(13), satisfy the boundary conditions

$$\tilde{R}_+ = S_+ = 0.$$  

(19)

It is easy to verify that, in terms of these new fields, the WI (15)–(16) decouple

$$\frac{2\pi}{k} \int du (\tilde{\partial} j_{\tilde{R}} + \partial j_{\tilde{R}}) = \frac{1}{\alpha_4 (1 - \alpha_2)} \tilde{\partial} R_+,$$

(20)

$$\frac{2\pi}{k} \int du (\tilde{\partial} j_{\tilde{S}} + \partial j_{\tilde{S}}) = \frac{1}{\alpha_4 (1 - \alpha_2)} \partial S_+,$$

(21)

from which we read the chirality conditions

$$\tilde{\partial} R_+ = \partial S_+ = 0.$$  

(22)

Note that $R_+$ and $S_+$ are related by T symmetry and, in virtue of (19) and (22), are conserved currents. From the WI (20)–(21), we immediately obtain the following (Abelian limit of a)
Kač–Moody algebra:
\[
[R_4(z), R_4(z')] = \frac{2\pi \alpha_4(1 - \alpha_2)}{k} \partial \delta(z - z'),
\]
\[
[\bar{S}_4(\bar{z}), \bar{S}_4(\bar{z}')] = \frac{2\pi \alpha_4(1 - \alpha_2)}{k} \partial \delta(\bar{z} - \bar{z}'),
\]
\[
[R_4(z), \bar{S}_4(\bar{z})] = 0,
\] (23)
which are connected by T and where the last commutator shows the decoupling of the two currents. We stress that, in the Abelian case, the central charge is not completely fixed since it depends on the boundary parameters. The additional requirement of decoupling of the boundary action (5) in terms of \(R_4\) and \(S_4\) fixes the values of the parameters \(\alpha_2 = 0\) and \(\alpha_4 = 1\).

The result of our analysis is the presence, on the boundary of the Abelian BF model, of two conserved currents with opposite chiralities, connected by T symmetry. The above picture is in agreement with the phenomenology involved in the helical edge states of the QSH effect [8, 9, 11], in agreement with the idea that the BF model is a good effective field theoretical description of the 2+1 D TI [17]. Note that, due to the presence of the factor \(k\), our results could be applied to possible fractional extension as well [15].

5. The non-Abelian BF model

The non-Abelian generalization of (1) is
\[
S_{\text{BF}} = \frac{k}{2\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \{ F_{\mu\nu}^a B_\rho^a + \frac{1}{2} f^{abc} B_\mu^a B_\nu^b B_\rho^c \},
\] (24)
where the fields \(A_\mu^a\) and \(B_\mu^a\) belong to the adjoint representation of a compact simple gauge group \(G\) whose structure constants are \(f^{abc}\). In (24), the non-Abelian field strength is defined as \(F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c\) and the coupling \(k\) can be related to the mutual statistics between the quasiparticle sources of the fields. It is worth noting that \(k\) can be connected to the ‘cosmological constant’ usually defined in the literature [24].

In the light-cone coordinates, the action in (24) reads
\[
S_{\text{BF}} = \frac{k}{\pi} \int du \, d^2z \{ B^a (\partial A^a_\mu - \partial_\mu A^a + f^{abc} A^b \bar{A}^c) + \bar{B}^a (\partial_\mu A^a - \partial A^a_\mu + f^{abc} A^b A^c) + B^a (\partial \bar{A}^a - \bar{A}^a + f^{abc} \bar{A}^b \bar{A}^c) + \bar{B}^a (\partial A^a - A^a + f^{abc} A^b A^c) \}.
\] (25)

The axial gauge \(A^a_\mu = B^a_\mu = 0\) is realized, as in the Abelian case, through a suitable gauge fixing term which involves ghost, antighost and Lagrange multipliers for both the gauge fields \(A^a_\mu\) and \(B^a_\mu\). It is straightforward to verify that the above action satisfies T and P symmetries. The non-Abelian gauge symmetry is realized, as is usual in the axial gauge, by means of the local WI \(H^a[Z_\mu] = N^a[Z_\mu] = 0\), where \(H^a(u, z, \bar{z})\) and \(N^a(u, z, \bar{z})\) are local operators whose explicit form is inessential here, and can be found in [24], where the non-Abelian BF theory in the axial gauge with and without boundary is treated in greater detail.

As in the Abelian case, the introduction of the planar boundary \(u = 0\) is realized through the addition of a term in the action or, equivalently, through \(\delta(u)\)—breakings of the field equations, satisfying the constraints of consistency, power counting and helicity conservation.
Consequently, the local WI are broken by the boundary and, once integrated over the \( u \)-coordinate, read
\[
\frac{\pi}{k} \int du \ H^a[Z_u] = \Delta^a_{\mu}(z, \bar{z}),
\]
\[
\frac{\pi}{k} \int du \ N^a[Z_u] = \Delta^a_{\nu}(z, \bar{z}).
\]
(26)
(27)

The above equations are the non-Abelian counterpart of (15)–(16) and likewise they describe the residual gauge invariance left on the boundary. By choosing the parameters \( \alpha_1 = \alpha_4 \) and \( \alpha_2 = \alpha_3 \) the terms \( \Delta^a_{\mu} \) and \( \Delta^a_{\nu} \) on their rhs are linear in the quantum fields (hence classical and allowed). Note that the WI (26)–(27) are the most general form of gauge invariance for the non-Abelian theory with boundary [27]. This is a central point for what follows. Requiring T symmetry at the boundary and (14), still valid in the non-Abelian case, one fixes the parameters of the boundary action to be \( \alpha_1 = \alpha_4 = 1 \) and \( \alpha_2 = \alpha_3 = 0 \).

The most general conserved chiral currents on the boundary, together with their algebra, can be fully determined [24]. It turns out that the only T symmetric solution is given by a direct sum of two independent Kač–Moody algebras satisfied by conserved currents of opposite chirality.

In more detail, the WI corresponding to this solution read
\[
\frac{\pi}{k} \int du \ H^a[Z_u] = - (\partial \bar{A}^a_z + \bar{\partial} A^a_z),
\]
\[
\frac{\pi}{k} \int du \ N^a[Z_u] = - (\partial \bar{B}^a_z + \bar{\partial} B^a_z).
\]
(28)
(29)

The linear combinations of fields
\[
R^a_z \equiv A^a_z + B^a_z, \quad S^a_z \equiv - A^a_z + B^a_z
\]
(30)
satisfy the Dirichlet boundary conditions \( S^a_z(z, \bar{z}) = \bar{R}^a_z(z, \bar{z}) = 0 \), and the residual WI (28)–(29) identify two conserved currents with opposite chiralities \( \bar{\partial} R^a_z(z, \bar{z}) = \partial \bar{S}^a_z(z, \bar{z}) = 0 \), for which the following direct sum of Kač–Moody algebras living on the same side of the plane \( u = 0 \) with opposite chiralities holds:
\[
\begin{align*}
[R^a_z(z), \ R^b_{z'}(z')] &= f^{abc} \delta(z - z') R^c_z(z) + \frac{2\pi}{k} \delta^{ab} \delta(z - z'), \\
[\bar{S}^a_z(\bar{z}), \ \bar{S}^b_{\bar{z}'}(\bar{z}')] &= f^{abc} \delta(\bar{z} - \bar{z'}) \bar{S}^c_z(\bar{z}) + \frac{2\pi}{k} \delta^{ab} \delta(\bar{z} - \bar{z'}), \\
[R^a_z(z), \ \bar{S}^b_{\bar{z}'}(\bar{z}')] &= 0,
\end{align*}
\]
(31)
where the last commutator shows that \( R_z \) and \( \bar{S}_z \) are decoupled. Note that, in the Abelian limit, the above WI and algebras reduce to (15)–(16) and (23), respectively, in the case of decoupling of the boundary action.

Finally, we note that, as has been shown in [28], the redefinition (30) allows us to rephrase the 2 + 1 D non-Abelian BF theory in (24) in terms of two CS theories with opposite coupling constants in the bulk. As stated before, in the non-Abelian case, the consistency of the theory also requires complete decoupling at the boundary that, conversely, is not needed in the Abelian case.

Note that other solutions exist that break T symmetry at the boundary, but we will not consider them here.
6. Conclusions

We investigated the 2 + 1 D BF model in the presence of a boundary both in the Abelian and in the non-Abelian case as a proper effective field theory for the QSH states. The key points of our analysis have been the combined application of Symanzik’s separability condition and the T symmetry. By means of these conditions we have been able to evaluate the algebraic structure of the current at the edge of the system. In the Abelian case we found two Kać–Moody current algebras with opposite chiralities and with the central charges depending on the coupling constant \( k \), together with arbitrary boundary parameters. In the non-Abelian case we also find two Kać–Moody currents with opposite chiralities with the important difference that the central charge is unambiguously determined in terms of the coupling constant \( k \) of the theory. The appearance of Kać–Moody algebras related by T symmetry reflects the equivalence between the double CS model and the non-Abelian BF model once the two CS are completely decoupled at the boundary [28]. The QSH is characterized by the presence of currents with opposite spin and chiralities on the boundary [9–11]. Indeed, one of the new results of this paper is that we find an algebraic structure which displays counter-propagating currents along the same edge and, in addition, it respects the T invariance.

Symanzik’s method of treating boundaries in field theory is of general applicability and is model independent. It is precisely for this reason that we adopted it, having in mind two recently and widely discussed generalizations: the 3 + 1 D case and the breaking of the T symmetry, which can be easily treated along the same lines illustrated in this paper.

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