Non-Markovian dynamics in the theory of full counting statistics

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Abstract. We consider the theoretical description of real-time counting of electrons tunneling through a Coulomb-blockade quantum dot using a detector with finite bandwidth. By tracing out the quantum dot we find that the dynamics of the detector effectively is non-Markovian. We calculate the cumulant generating function corresponding to the resulting non-Markovian rate equation and find that the measured current cumulants behave significantly differently compared to those of a Markovian transport process. Our findings provide a novel interpretation of noise suppression found in a number of systems.

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The theory of full counting statistics concerns the probability \( P(n,t) \) of having transferred \( n \) charges through a mesoscopic system at time \( t \), when starting counting at \( t = 0 \) [1]. Rather than the probability distribution \( P(n,t) \), it is often more convenient to consider the cumulant generating function \( S(\chi,\lambda) \) defined as

\[
e^{S(\chi,t)} \equiv P(\chi,t) = \sum_n P(n,t)e^{i\lambda n},
\]

from which the zero-frequency cumulants of the current can be found in the long-\( t \) limit by deriving with respect to the counting field \( \chi \) at zero, i.e.,

\[
\left\langle (t^n) \right\rangle = \left. \frac{d^n S(\chi,\lambda)}{dt^n} \right|_{\lambda=0,t=\infty}, \quad n = 1, 2, 3, \ldots
\]

In this work we consider the effects of a finite bandwidth of the apparatus detecting charge transfers on the measured counting statistics. In particular, we show that the finite bandwidth makes the effective dynamics of the detector non-Markovian, and we discuss how non-Markovian dynamics in general can make the counting statistics and the corresponding current cumulants behave significantly differently compared to Markovian transport processes. Although, the conclusions reached below are obtained for a specific setup, we argue that they are valid for a large class of systems.

We consider a model of real-time counting with a finite-bandwidth detector [2] recently employed in order to explain experimental counting statistics results on electron transport through a Coulomb-blockade quantum dot [3]. In the experiment a quantum
point contact was used to monitor the two charge states participating in transport through a nearby quantum dot weakly coupled to source and drain electrodes and each switching event between the two charge states was associated with an electron either entering the quantum dot from the source electrode or leaving it via the drain. Rather than just considering the two charge states of the quantum dot, while keeping track of the number of electrons $n$ that have tunneled through the quantum dot, the model also takes into account the state of the detector that counts the electrons. In the following $P_i(n,t)$ denotes the probability that the system at time $t$ is in a state, where the quantum dot is occupied by $i = 0, 1$ extra electrons, while the detector indicates $j = 0, 1$ extra electrons on the quantum dot, and $n$ electrons according to the detector have been transferred through the quantum dot. We collect these four probabilities in the vector $\mathbf{P} = (P_0, P_1, P_1^*, P_0^*)^T$ and note that $P(n,t) = P_0(n,t) + P_1(n,t) + P_1^*(n,t) + P_0^*(n,t)$. The counting field is now introduced via a Fourier transformation as in Eq. (1) and the Markovian equation of motion for $\mathbf{P}(\chi, t)$ then reads

$$
\frac{d}{dt} \mathbf{P}(\chi, t) = \mathbf{M}(\chi) \mathbf{P}(\chi, t),
$$

where

$$
\mathbf{M}(\chi) = \begin{pmatrix}
-\Gamma_L & \Gamma_R & 0 & \Gamma_D \\
\Gamma_L & -(\Gamma_D + \Gamma_R) & 0 & 0 \\
0 & \Gamma_D e^{i\chi} & -\Gamma_R & \Gamma_L \\
0 & 0 & \Gamma_R & -(\Gamma_D + \Gamma_L)
\end{pmatrix}.
$$

Here, $\Gamma_L$ and $\Gamma_R$ denote the rates at which electrons are injected and leave the quantum dot, respectively, while $\Gamma_D$ is the rate (or the bandwidth) at which the detector reacts to changes of the charge state of the quantum dot (see Fig. 1). An ideal detector ($\Gamma_D \to \infty$) is able to count every electron that is transported through the quantum dot. On the other hand, when $\Gamma_D$ is comparable to the electron tunneling rates $\Gamma_L$ and $\Gamma_R$, the finite bandwidth of the detector reduces the ability of the detector to count every electron transfer event. This, of course, affects the measured counting statistics.
In the following, we trace out the quantum dot and show that the resulting dynamics of the detector is non-Markovian. The quantum dot is traced out by defining \( P_j \equiv \sum_{i=0}^{1} R_{ij} \), \( j = 0, 1 \), whose equations of motion read \( \dot{P}_0 = \Gamma_D (P_{01} - P_{00}) \) and \( \dot{P}_1 = \Gamma_D (\dot{\mathcal{E}} P_{01} - P_{01}) \), respectively. The probabilities \( P_j(\chi, t), j = 0, 1 \), only contain information about the state of the detector and the number of electrons counted by the detector. By observing that \( \dot{P}_{00} = \Gamma_L P_{00} - (\Gamma_D + \Gamma_R) P_{01} = \Gamma_L P_0 - (\Gamma_D + \Gamma_L + \Gamma_R) P_{01} \) and \( \dot{P}_{01} = \Gamma_R P_{11} - (\Gamma_D + \Gamma_L + \Gamma_R) P_{01} \), we find
\[
P_{01}(\chi, t) = \Gamma_L \int_0^t d\tau e^{-(\Gamma_D + \Gamma_L + \Gamma_R)(t-\tau)} P_0(\chi, \tau) + e^{-(\Gamma_D + \Gamma_L + \Gamma_R)\tau} P_{01}(\chi, t = 0),
\]
and a similar expression for \( P_{01}(\chi, t) \). In the following, we focus on the long-\( t \) limit, where the initial condition \( P_{10}(\chi, t = 0) \) (and \( P_{01}(\chi, t = 0) \)) may safely be neglected.\(^1\)

This leads to a non-Markovian rate-equation for \( p(\chi, t) = (P_0, P_1)^T \), reading
\[
\frac{d}{dt}p(\chi, t) = \int_0^t W(\chi, t - \tau)p(\chi, \tau),
\]
where
\[
W(\chi, t - \tau) = \Gamma_D e^{-(\Gamma_D + \Gamma_L + \Gamma_R)(t-\tau)} \begin{pmatrix} -\Gamma_L & \Gamma_R \\ \Gamma_L e^{\mathcal{E} \chi} & -\Gamma_R \end{pmatrix}. \tag{7}
\]
In Laplace space this translates to the algebraic equation
\[
\mathcal{W}(\chi, z) - p(\chi, t = 0) = W(\chi, z)p(\chi, z) \tag{8}
\]
or
\[
p(\chi, z) = \frac{1}{z - W(\chi, z)} p(\chi, t = 0) \tag{9}
\]
with
\[
W(\chi, z) = D(z) \begin{pmatrix} -\Gamma_L & \Gamma_R \\ \Gamma_L e^{\mathcal{E} \chi} & -\Gamma_R \end{pmatrix}, \tag{10}
\]
having introduced \( D(z) = \Gamma_D / (z + \Gamma_D + \Gamma_L + \Gamma_R) \). We note that in the limit \( \Gamma_D \to \infty \), \( D(z) \to 1 \), and the detector follows the Markovian dynamics of the quantum dot.

One can show (see e.g. Refs. [4, 5]) that the cumulant generating function in the long-\( t \) limit is given as \( S(\chi, t) = z^* (\chi) t \), where \( z^* (\chi) \) solves the equation
\[
z^* (\chi) - \Lambda_0 [\chi, z^* (\chi)] = 0. \tag{11}
\]
Here \( \Lambda_0 [\chi, z] \) is the eigenvalue of \( \mathcal{W}(\chi, z) \) which for \( \chi = 0 \) is zero, i.e., \( \Lambda_0 [0, z] = 0 \), and the solution \( z^* (\chi) \) must be chosen such that \( z^* (0) = 0 \). We find \( \Lambda_0 [\chi, z] = D(z) \lambda_0 (\chi) \) with \( \lambda_0 (\chi) = - (\Gamma_L + \Gamma_R) / 2 + \sqrt{(\Gamma_L + \Gamma_R)^2 / 4 + \Gamma_L \Gamma_R (e^{\mathcal{E} \chi} - 1)} \), and
\[
z^* (\chi) = - \frac{\Gamma_D + \Gamma_L + \Gamma_R}{2} + \sqrt{\left( \frac{\Gamma_D + \Gamma_L + \Gamma_R}{2} \right)^2 + \Gamma_D \lambda_0 (\chi)}. \tag{12}
\]

\(^1\) We note that the initial condition plays a crucial role when studying finite-frequency fluctuations.
For large matrices, in general, it may be non-trivial to find $z^\ast (\chi)$ and more sophisticated methods, as the one we describe in Ref. [5], may be needed. Having found the cumulant generating function in the long-$t$ limit, $S(\chi, t) = z^\ast (\chi)t$, we may calculate the current cumulants, and here we just give the results for the first two current cumulants, although it, in principle, is possible to obtain any cumulant having found $S(\chi, t)$,

$$
\langle \langle I^1 \rangle \rangle = \Gamma_R \left[ \frac{1+a}{2} \right] \times \left[ \frac{k}{1+k} \right],
$$

$$
\langle \langle I^2 \rangle \rangle = \left[ \frac{1+a^2}{2} - \frac{k(1-a^2)}{2(1+k)^2} \right] \langle \langle I^1 \rangle \rangle .
$$

(13)

These are the results also found in Ref. [3] using the model in its Markovian formulation given by Eq. (4), and following that work we have also introduced the asymmetry $a = (\Gamma_R - \Gamma_L)/(\Gamma_R + \Gamma_L)$ and the relative bandwidth $k = \Gamma_D/(\Gamma_R + \Gamma_L)$.

It is interesting to consider the so-called Fano factor $F \equiv \langle \langle \hat{I}^2 \rangle \rangle / \langle \langle \hat{I}^1 \rangle \rangle$. In the ideal detector limit $\Gamma_D \to \infty$, we find the well-known result $F = (1+a^2)/2$ for a Markovian two-state model with uni-directional transport where $1/2 \leq F \leq 1$. For finite bandwidths, the Fano factor may, however, be suppressed below $1/2$, and for the given model, we find that the Fano factor is bounded from below by the value $3/8$ ($a = 0, k = 1$). In a number of papers, the sensitivity of the counting statistics to coherent versus sequential tunneling has been discussed [6], and particularly, it has been conjectured that a suppression of the Fano factor below $1/2$ for transport through a double barrier resonant diode could be an indication of coherent tunneling rather than sequential [7]. The results found in the present work show that a suppression below $1/2$ can occur due to non-Markovian dynamics, which is not necessarily induced by quantum coherence, but in general arises from tracing out parts of a system. We believe that a similar interpretation can explain the recently calculated Fano factor suppression of incoherent transport through a single electron transistor (SET) coupled to a nano-mechanical resonator [8]. There, we believe that the dynamics of the SET effectively is non-Markovian due to the coupling to the resonator, which in turn can explain the suppression of the Fano factor below $1/2$.

In conclusion, we have presented a study of the full counting statistics of electron transport through a Coulomb-blockade quantum dot as measured by a detector with finite bandwidth. In particular, we have calculated the current cumulants of the measured charge transport described by a non-Markovian rate equation obtained by tracing out the quantum dot and only considering the dynamics of the detector. Our results show that non-Markovian effects may strongly effect the charge transport statistics.

REFERENCES

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