Full Counting Statistics in Strongly Interacting Systems: Non-Markovian Effects

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We present a theory of full counting statistics for electron transport through interacting electron systems with non-Markovian dynamics. We illustrate our approach for transport through a single-level quantum dot and a metallic single-electron transistor to second order in the tunnel coupling, and discuss under which circumstances non-Markovian effects appear in the transport properties.

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The study of current fluctuations in mesoscopic systems has become an intense field of research, since it allows us to access information about electron correlations that is not contained in the average current. The phenomenon of shot noise [1] that dominates the current noise at low temperatures has been investigated theoretically and experimentally in various contexts. Further understanding in electron transport can be gained from the study of higher moments of the current fluctuations. As a consequence of non-Markovian effects, the full counting statistics for strongly interacting systems with non-Markovian dynamics can strongly influence quantum transport. The effect of electron correlation on FCS has been considered so far in the case of a weakly interacting mesoscopic conductor [11], almost open dots [12], and charge shuttles [13]. The FCS for Coulomb-blockade devices has been analyzed [14,15] in the framework of a Born-Markov master equation approach. The aim of the present Letter is to extend this idea to obtain a general theory of full counting statistics for strongly interacting systems with non-Markovian behavior. In particular, we formulate a perturbative non-Markovian expansion that allows for a systematic study of the relative importance of non-Markovian corrections. We demonstrate for the example of a single-level quantum dot with strong Coulomb interaction that non-Markovian effects become increasingly important for higher moments of the current fluctuations.

Full counting statistics.—Full information about all transport properties of a given system is contained in the probability distribution $P(N, t)$ that $N$ charges have passed through the system after time $t$. The cumulant generating function (CGF) is defined as $S(\chi)$ by

$$S(\chi) = -\ln \left[ \sum_{N=-\infty}^{\infty} e^{iN\chi} P(N, t) \right]. \quad (1)$$

where $\chi$ is the counting field. All cumulants of the current can be obtained from the generating function by performing derivatives with respect to the counting field $(\langle i \rangle)_{\chi} = -(i)^n (\langle i^n \rangle)_{\chi}/n!$. The first four cumulants are related to the average current, the (zero-frequency) current noise, the skewness, and the kurtosis.

In this work we consider systems with strong local interactions, such as electrons in a quantum dot, that are coupled to a reservoir of noninteracting degrees of freedom. In these situations transport properties can often be described in terms of few (local) degrees of freedom (the charge of the quantum dot in the previous example). It is then convenient to integrate out the noninteracting degrees of freedom to arrive at an effective description of the reduced system only. Let $p^a$ be the vector whose entries are the probabilities that the system is in a given charge state at the initial time $t = 0$. The time evolution of the system is described by a generalized master equation,

$$\frac{d}{dt} p(N, t) = \sum_{N'=-\infty}^{\infty} \int_0^t dt' W(N - N', t, t') \cdot p(N', t'), \quad (2)$$

where $p(N, t)$ is the vector of dot occupation probabilities under the condition that $N$ electrons have passed the system. The CGF is given by Eq. (1) where $P(N, t) = e^{\chi^t} p(N, t)$ with $\chi^t = (1, 1, \ldots, 1)$. The matrix $W(N - N', t, t')$ describes transitions during which $N - N'$ electrons are transferred. The counting field $\chi$ is introduced by Fourier transforming the master equation $p(\chi, t) = \sum_N \exp(iN\chi) p(N, t)$ and $W(\chi, t, t') = \sum_N \exp(iN\chi) W(N, t, t')$. In general, the kernels $W(\chi, t, t')$ are nonlocal in time. We consider the case in which there is no explicit time dependence of the systems parameters, so that $W(\chi, t - t')$ can be Laplace transformed, $W(\chi, z) = \int_0^\infty dt \exp(-z t) W(\chi, t)$. Usually, the dynamics of the system is characterized by a time scale of the individual transitions. We assume that $W(\chi, t)$ decays faster than any power of $t$, such that the derivatives $\partial_\chi^n W(\chi, z)$ exist for all $n$. 

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In the Markovian limit, \( W(\chi, t) \sim \delta(t) \), the system at time \( t \) depends only on the state of the system at the same time \( t \). The master equation, Eq. (2), is then solved by \( p(x, t) = \exp[\mathbf{W}(\chi, z = 0)t] \cdot \mathbf{p}^{\text{in}} \); i.e., only the \( z = 0 \) component of the transitions \( \mathbf{W}(\chi, z) \) are taken into account.

Our goal is to describe stationary transport properties in the presence of a memory of the system, described by the full \( z \) dependence of \( \mathbf{W}(\chi, z) \). To solve the master equation without making use of the Markovian approximation, we switch to the Laplace representation,

\[
p(x, z) - p^{\text{in}} = \mathbf{W}(\chi, z) \cdot p(x, z),
\]

which is solved by \( p(x, z) = \sum_{n=0}^{\infty} \mathbf{W}(\chi, z)^n / z^{n+1} p^{\text{in}} \). By assuming that the kernel \( \mathbf{W}(\chi, t) \) decays in time faster than any power law, we can define the Taylor series \( [\mathbf{W}(\chi, z)]^n = \sum_{m=0}^{\infty} \partial^m z \mathbf{W}(\chi, z)]^{m} |_{z=0} z^m/m! \) and substitute it in the previous solution for \( p(x, z) \). The longtime behavior of \( p(x, z) \) is determined by its poles in \( z \). The terms proportional to positive power of \( z \) give rise to terms which contribute only to the transient regime. They can be disregarded in the longtime evolution. By performing the inverse Laplace transformation we get

\[
p(x, t) = \sum_{n=0}^{\infty} \frac{\partial^n [\mathbf{W}(\chi, z)]^n e^{\mathbf{W}(\chi, z)}|_{z=0}}{n!} p^{\text{in}}(4)
\]

for large \( t \). To proceed, we perform a spectral decomposition of the matrix \( \mathbf{W}(\chi, z) \). For physical reasonable systems, all eigenvalues have a negative real part. As a consequence of the exponential function in Eq. (4), the longtime behavior will be dominated by the eigenvalue \( \lambda(\chi, z) \) with the smallest absolute value of the real part. Let \( q_0 \) and \( p_0 \) be the corresponding left and right eigenvectors, \( q_0^\dagger \cdot \mathbf{W}(\chi, z) = \lambda(\chi, z)q_0^\dagger \), and \( \mathbf{W}(\chi, z) \cdot p_0 = \lambda(\chi, z)p_0 \). Unitarity in the absence of counting fields implies \( \lambda(0, z) = 0 \) for all \( z \). The CGF becomes \( S(\chi) = -\ln [\sum_{n=0}^{\infty} \frac{1}{n!} \partial^n [\lambda^{\sum_{n=0}^{\infty} \frac{1}{n!} \partial^n (\lambda^{\sum_{n=0}^{\infty} \frac{1}{n!} \partial^n (a^{(n+1)}))}] |_{\lambda=0} \] with \( \lambda(\chi, z) = \ln [\mathbf{W}(\chi, z))] \). By performing the time derivative and making use of the relation \( \sum_{n=0}^{\infty} \frac{1}{n!} \partial^n (a^{(n+1)})/ \sum_{n=0}^{\infty} \frac{1}{n!} \partial^n (a^{(n)}) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial^n (b^{(n+1)})/ \sum_{n=0}^{\infty} \frac{1}{n!} \partial^n (b^{(n)}) \) that holds for arbitrary functions \( a \) and \( b \), we arrive at the central result of this Letter,

\[
S(\chi) = -\frac{\sum_{n=0}^{\infty} \frac{1}{n!} \partial^n [\sum_{n=0}^{\infty} \frac{1}{n!} \partial^n (\lambda^{n+1}(\chi, z))] |_{\lambda=0}}{\sum_{n=0}^{\infty} \frac{1}{n!} \partial^n (\lambda^{n}(\chi, z))} .
\]

The CGF depends only on the eigenvalue \( \lambda(\chi, z) \). This result can be used as a starting point for a non-Markovian expansion, \( S(\chi) = \sum_{n=0}^{\infty} S_n(\chi) \), where \( S_n(\chi) \) contains \( n \) derivatives with respect to \( \lambda \) of \( \lambda \) [16]. While \( S_0(\chi) \) describes the Markovian limit, \( S_n(\chi) \) is the \( n \)th non-Markovian correction [17].

Perturbative non-Markovian expansion.—For many systems there is a small parameter which allows for a perturbative analysis of all transport properties. In the examples to be discussed below this will be the tunnel-coupling strength between the leads and the interacting region (quantum dot or metallic island). Then, \( \lambda(\chi, z) = \sum_{i=1}^{\infty} \lambda_i(\chi, z) \), where the superscript \( (i) \) indicates the order in the small parameter. The lowest-order transport properties are derived from the lowest-order CGF \( S^{(1)}(\chi) = -i \lambda^{(1)}(\chi, z)|_{z=0} \), as found in Ref. [14]. Non-Markovian corrections, signaled by derivatives \( \partial^2 \lambda(\chi, z) \), do not enter in this limit. The highest order term of the non-Markovian expansion that enters the evaluation of the \( m \)th cumulant, in \( m \)th order perturbation theory, is the 4th term with \( k = m \{ m, m \} - 1 \). As a consequence, non-Markovian behavior is probed only in the second or higher cumulant combined with second or higher order in perturbation theory [18]. The second-order contribution, for example, reads \( S^{(2)}(\chi) = -i \lambda^{(2)}(\chi, z) + \lambda^{(1)}(\chi, z)\partial \lambda^{(1)}(\chi, z)|_{z=0} \). The appearance of these derivatives in the noise of second-order transport through quantum dots has been also found in Ref. [19]. In the remaining part of this Letter, we illustrate our approach with two examples. We calculate the CGF for second-order transport through a single-level quantum dot and through a metallic single-electron transistor in the presence of strong Coulomb interaction.

Single-level QD.—The single-level quantum dot is described by the Hamiltonian, \( H = H_L + H_R + H_D + H_T \). The electrons in the noninteracting left and right leads are represented by \( H_L \) and \( H_R \), respectively; \( H_D = e_s \sigma_s \sigma + U n_1 n_1 \) describes the dot with level energy \( e \) and charging energy \( U \) for double occupation. Tunneling is modeled by \( H_{T;r} = \sigma_s \sigma + \text{H.c.} \) with \( r = R, L \), where we assume the tunnel matrix element \( t_r \) to be independent of momentum \( k \) and spin \( \sigma \). The tunnel-coupling strength is characterized by the intrinsic linewidth \( \Gamma = \Gamma_L + \Gamma_R \) with \( \Gamma_r = 2 \rho_r |\chi_r|^2 \) where \( \rho_r \) is the density of states in the leads. An asymmetry of the tunnel couplings is parametrized by \( \gamma = 4 \Gamma_L \Gamma_R / |\chi_L|^2 / |\chi_R|^2 \). To derive the kernels \( \mathbf{W} \) of the generalized master equation, we make use of a diagrammatic real-time technique for the time evolution of the reduced density matrix formulated on a Keldysh contour as described in Ref. [20]. We introduce counting fields \( \chi_r \) for tunneling through barrier \( r \) into lead \( r \) by the replacement \( t_r \rightarrow t_r \exp(i \chi_r) \) for tunnel vertices on the upper and \( t_r \rightarrow t_r \exp(-i \chi_r) \) on the lower branch of the Keldysh contour with \( \chi_L = -\chi_R = \chi / 2 \).

We consider the limit \( U \rightarrow \infty \), in which double occupancy of the dot is prohibited [21], and obtain

\[
S^{(1)}(\chi) = \frac{i \Gamma \tilde{f}(\chi)}{2 \hbar} \left[ 1 - \frac{1 + 2 \gamma \sum_k f_{\chi}(\epsilon_k)(e^{i \chi_k} - 1)}{\tilde{f}(\chi)} \right]
\]

with \( f_{\chi}(\omega) = [1 - f_L(\omega)]f_R(\omega) \), \( f_{\gamma}(\omega) = f_L(\omega)[1 - f_R(\omega)] \), and \( \tilde{f}(\chi) = \sum_k \Gamma_k [1 + f_{\chi}(\omega)] / \Gamma_k \), where \( f_{\chi}(\omega) \) is the Fermi function for lead \( r \). This result was previously obtained in [14]. The second-order contribution, \( S^{(2)}(\chi) = S^{(2)}_{\text{col}}(\chi) + S^{(2)}_{\text{rec}}(\chi) \), consists of two terms.
The first one,
\[ S^{(2)}_{\text{cot}}(\chi) = -\frac{t\gamma^2}{4\pi\hbar} \sum_{k=\pm} (e^{ikx} - 1) \int d\omega f_{l,k}(\omega) R(\omega - \epsilon), \]

(7)

with \( R(\omega) = \text{Re}[1/(\omega + i0^+)^2] \), describes cotunneling processes [22], and is in agreement with previous work about noise in cotunneling regime [23]. The counting-field dependence corresponds to a bidirectional Poisson statistics of a single barrier where the transition rates are substituted by the cotunneling rates of the quantum dot. However, \( S^{(2)}(\chi) \) contains a second contribution,
\[ S^{(2)}_{\text{ren}}(\chi) = \partial_{\epsilon} [S^{(1)}(\chi) \text{Re}[\sigma(\epsilon)]] \]

(8)

with \( \sigma(\epsilon) = -\sum_{r} (\Gamma_r/2\pi) \int d\omega f_{l,k}(\omega)/(\omega - \epsilon + i0^+) \). In the previous formulas a high-energy cutoff \( E_c \), of the order of charging energy, has to be introduced in order to cure spurious divergences related to the fact that we restricted the charge states to 0, 1 [24]. The contribution \( S^{(2)}_{\text{ren}}(\chi) \) is also of second order in the tunnel-coupling strength but obeys the same statistics as the first-order (sequential-tunneling) result, Eq. (6). This suggests that there are two different types of second-order contributions to transport. In addition to the usual cotunneling processes, there are corrections to sequential tunneling due to quantum-fluctuation induced renormalization of the system parameters [25]. From the form of Eq. (8) we deduce a renormalization of level position and coupling strength given by
\[ \bar{\epsilon} = \epsilon + \text{Re} \sigma(\epsilon) \quad \text{and} \quad \bar{\Gamma} = \Gamma [1 + \partial_{\epsilon} \text{Re} \sigma(\epsilon)]. \]

In Fig. 1 we plot the first four cumulants as a function of level position \( \epsilon \). The solid lines represent the full first- plus second-order result, as compared to the first-order contribution (dashed line). The relative importance of the non-Markovian contributions is illustrated in Fig. 2. While for the current only Markovian contributions enter (see discussion above), non-Markovian corrections become increasingly important for higher cumulants.

**Metallic QD.**—A similar analysis can be performed for a metallic single-electron transistor, which accommodates a continuum of states on the dot and includes a large number of transverse channels. Following the notation of Ref. [24], we characterize the tunnel-coupling strength by the dimensionless conductance \( \alpha_0^r = \hbar/(4\pi e^2 R_r) \) where \( R_r \) is the resistance of barrier \( r = L, R \). We concentrate on the low-temperature regime, in which only two charge states of the metallic island have to be taken into account. This requires, again, the introduction of an high-energy cutoff to regularize the integrals. The difference of the charging energy between them is denoted by \( \Delta \). We obtain for the first-order CGF
\[ S^{(1)}(\chi) = \frac{t\pi \alpha(\Delta)}{\hbar} \sqrt{1 + 4 \sum_{k=\pm} \frac{\alpha(k_0)(\Delta)(e^{ikx} - 1)}{\alpha(\Delta)}}. \]

(9)

where we have used the definitions \( \alpha(\omega) = \sum_{r=L,R} \alpha_r(\omega) \) and \( \alpha(\pm)(\omega) = \alpha_{\pm}^L(\omega) \alpha_{\pm}^R(\omega) \) with \( \alpha_{\pm}^L(\omega) = \alpha_r(\omega)f_{\pm}^r(\omega) \) where \( f_{\pm}^r(\omega) \) are the Fermi functions of lead \( r \) and \( \alpha_r(\omega) = \alpha_0^r \coth[\beta(\omega - \mu_r)/2] \). The second-order term is given by \( S^{(2)}(\chi) = S^{(2)}_{\text{cot}}(\chi) + S^{(2)}_{\text{ren}}(\chi) \), as the case of the single-level quantum dot. Again, there is the cotunneling term.

**FIG. 1.** The first (a), second (b), third (c), and fourth (d) cumulant for a single-level quantum dot with \( U \to \infty \) as a function of the level position \( \epsilon/eV_{sd} \) for first (dashed lines) and first- plus second-order (solid lines) in tunneling with \( eV_{sd} = (\mu_R - \mu_L) \). Other parameters are \( \gamma = 1, \Gamma = 3 \times 10^{-2}eV_{sd}, k_BT = 10^{-1}eV_{sd}, \) and the high-energy cutoff \( E_c = 10eV_{sd} \).

**FIG. 2.** Relative contribution of the non-Markovian part \( \langle I \rangle_n - \langle I \rangle_n^{\text{Markov}} \) to the \( n \)th cumulant \( \langle I \rangle_n \) in first- plus second-order in tunneling. The full, dashed, and dot-dashed lines correspond to the second, third, and fourth cumulant for the same parameters as in Fig. 1.
\[ S^\text{(2)}_{\text{col}}(\chi) = -\frac{2 \pi i}{\hbar} \sum_{k=\pm} (e^{ikx} - 1) \int d\omega \alpha_{(k)}(\omega) R(\omega - \Delta), \tag{10} \]

and \[ S^\text{(2)}_{\text{int}}(\chi) = \partial_\Delta [S^\text{(1)}(\chi) \text{Re}[\sigma(\Delta)]] \]

associated with sequential-tunneling processes with renormalized system parameters, where \[ \sigma(\Delta) = -\int d\omega \alpha(\omega)/(\omega - \Delta + i0^+). \]

This interpretation of the different types of second-order contributions is consistent with the analysis of the second-order current \cite{26} and of the FCS within a drone Majorana fermion representation \cite{27}.

**Conclusions.**—We present a theory of FCS for interacting systems with non-Markovian dynamics. A general expression for the CGF is derived that provides the starting point for a perturbative non-Markovian expansion. As examples we study transport through a single-level quantum dot and a metallic single-electron transistor to second order in the tunnel-coupling strength. From our formulation we could identify two different types of contributions to second-order transport, namely, cotunneling and corrections to sequential tunneling due to renormalization of the system parameters. Furthermore, we demonstrate the increasing importance of non-Markovian effects for higher cumulants and higher orders in the tunnel-coupling strength. We thank D. Bagrets, W. Belzig, Y. Gefen, M. Hettler, G. Johansson, F. Plastina, A. Romito, M. Sassetti, and A. Thielmann for useful discussions. Financial support from DFG via SFB 491 and GRK 726, EU-RTN-RTNNANO, EU-RTN-Spintronics, MIUR-Firb is gratefully acknowledged.


[16] The non-Markovian expansion can also be generated with the iterative equation \[ S^\text{ren}_{+1}(\chi) = -\mathcal{R}(\chi)[\lambda^{\text{ren}+2}(\chi)]/(n + 1) - \sum\lim_{\Delta \to 0} S^\text{ren}_{\Delta}(\chi) \delta^{\text{ren}}[\lambda^{\text{ren}}(\chi)]/(n - i)! |_{\Delta \to 0}. \]

[17] From Eq. (4) one can show that the generating function can be obtained as a solution of the equation \[ z^* - \lambda(\chi, z^*) = 0. \]

We test that for the opposite limit, \[ U = 0 \], our theory is consistent with the exact CGF obtained from Levitov’s formula \cite{14} and expanded up to second order on \( \Gamma \).


[21] We checked that for the opposite limit, \( U = 0 \), our theory is consistent with the exact CGF obtained from Levitov’s formula \cite{14} and expanded up to second order on \( \Gamma \).


[25] The energy derivatives of Eq. (8) are not solely due to the non-Markovian corrections [which are both contained in \( S_{\text{col}}^\text{(ren)}(\chi) \) and \( S_{\text{int}}^\text{(ren)}(\chi) \)].
