Fidelity optimization for holonomic quantum gates in dissipative environments

Daniele Parodi, Maura Sassetti, Paolo Solinas, Paolo Zanardi, and Nino Zanghí
1Dipartimento di Fisica, Università di Genova, Via Dodecaneso 33, 16146 Genova, Italy
2Istituto Nazionale di Fisica Nucleare (Sezione di Genova), Via Dodecaneso 33, 16146 Genova, Italy
3INFN-CNR Lamia, Via Dodecaneso 33, 16146 Genova, Italy
4Institute for Scientific Interchange, Viale Settimo Severo 65, 10133 Torino, Italy

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We analyze the performance of holonomic quantum gates in semiconductor quantum dots, driven by ultrafast lasers, under the effect of a dissipative environment. The environment is modeled as a thermal bath of oscillators linearly coupled with the electron states of the quantum dot. Standard techniques make the problem amenable to a numerical treatment and allow one to determine the fidelity as a function of all the relevant physical parameters. As a consequence of our analysis, we show that the disturbance of the environment can be (approximately) suppressed and the performance of the gate optimized—provided that the thermal bath is purely super-Ohmic. We conclude by showing that such an optimization is impossible for Ohmic environments.

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I. INTRODUCTION

In the last years holonomic quantum computation (HQC) has proved to be a viable and promising approach to quantum information processing and quantum computation [1]. According to this approach, quantum information is encoded in an n-fold degenerate eigenspace of a family of quantum Hamiltonians depending on dynamically controllable parameters. Recently, concrete proposals for quantum computation have been put forward, for both Abelian [2] and non-Abelian holonomies [3,4]. The so-constructed gates depend only on the global geometrical feature—e.g. the solid angle spanned in the parameter space—and because of this, it is believed that they are robust against errors affecting the physical parameters controlling the gates themselves (e.g., laser pulses). This expectation has been confirmed by recent investigations [5].

An important open problem is whether holonomic quantum gates are stable under the effect of the environment [6]. In this paper we argue that holonomic gates have indeed a good performance when the effect of the environment is taken into account. We show this on the basis of a simple and idealized model which covers situations describing electron states in quantum dots ranging from excitons [4] to optically active spin-degenerate ones [7] (for a related investigation about environmental effects on semiconductor-based quantum gates see Ref. [8]). For such gates, the main source of dissipation is due to phonons with super-Ohmic spectral density [9]. It turns out that these gates manifest a rich structure when the “control parameters” are changed. By varying in a suitable way the adiabatic time, the super-Ohmic effect can be minimized and suppressed. For completeness, we have extended our analysis to an Ohmic environment.

In Sec. II, we review the HQC model, introduce a model for the dissipative environment, and write the master equation to solve. In Sec. III, we describe the computer simulations and specify the kind of environment we consider. From the numerical results we deduce nontrivial behavior of the fidelity (the gate performance estimator), which allows us to minimize the decoherence effects of a super-Ohmic environment. We show that analogous minimization cannot be done with other kinds of environment (Ohmic).

II. HQC IN A DISSIPATIVE ENVIRONMENT

The physical system we consider is constituted by three degenerate (or quasidegenerate) states (|+, |−, and |0⟩) optically connected to another state |G⟩. Every degenerate state is separately addressed by polarization or frequency selection with a laser. This model describes various quantum systems interacting with a laser radiation field; here, we deal with quantum states in semiconductor quantum dots such as excitons [4] and spin-degenerate electron states [7].

The (approximate) Holonomic Hamiltonian modeling the effect of the laser on the system is (for simplicity, \( \hbar = 1 \)) [4]

\[
H_0(t) = \sum_{j=+,-,0} [\epsilon_j |j\rangle\langle j| + (e^{-i\Omega_j(t)} |j\rangle\langle G| + H.c.)],
\]

where \( \Omega_j(t) \) are the time-dependent Rabi frequencies depending on controllable parameters, such as the phase and intensity of the lasers, and \( \epsilon \) is the energy of the degenerate electron states. The Rabi frequencies can be modulated within the adiabatic time \( t_{ad} \) (which coincides with the gating time) to produce a loop in the parameter space and thereby realize the periodic condition \( H_0(t_{ad}) = H_0(0) \).

It should be observed that different loops in the parameter space produce different holonomic operators. Here we consider two different sets of Rabi frequencies—that is, two different choices of the time-dependent functions \( \Omega_j(t) \) in Eq. (1). According to the holonomic approach, the corresponding unitary evolutions occurring in the adiabatic time define the unitary transformations associated with the holonomic quantum gates [4]. We shall hereafter refer to the gates defined by these two different sets of Rabi frequencies (which form a complete set of single qubit gates) as “gate 1” and “gate 2”; they correspond, respectively,
to the unitary operators $U_1 = e^{i(\pi/4)|+\rangle\langle+|}$ and $U_2 = e^{i(\pi/2)\sigma_y}$ [where $\sigma_y = i(|+\rangle\langle-| - |\rangle\langle+|)$].

The Hamiltonian (1) has four eigenstates: two eigenstates $|B_\pm\rangle$ with time-dependent eigenvalues $\epsilon_n$ (called bright states) and two eigenstates $|D_{1,2}\rangle$ with constant and degenerate eigenvalue $\epsilon$ (called dark states). To construct a complete set of holonomic quantum gates it is sufficient to restrict Rabi frequencies $\Omega(t)$ such that the norm $\Omega$ of the vectors $\{\Omega_j(t)\}$ is time independent. Under this condition, it can be easily shown that the two dark states have energy $\epsilon$ and the two bright states have time-independent energies

$$\epsilon_n = (\epsilon \pm \sqrt{\epsilon^2 + 4\Omega^2})/2.$$  

(2)

The adiabatic condition is simply $\Omega_{\text{tad}} \gg 1$.

The environment is described as an ensemble of harmonic oscillators linearly coupled to the system [10], with total Hamiltonian

$$H = H_0(t) + \sum_{a=1}^{N} \left( \frac{\hat{p}_a^2}{2m_a} + \frac{1}{2} m_a \omega_a^2 \hat{x}_a^2 + c_{\alpha} \hat{x}_a \right).$$  

(3)

The interaction should break the degeneracy of the degenerate states, a condition that is easily fulfilled by assuming the operator $A$ to be of the form $A = \text{diag}(0,1,0,-1)$ in the basis $|G\rangle, |+\rangle, |0\rangle$, and $|-\rangle$. [Equation (3) includes implicitly the standard renormalization term $|9|$.]

We now consider the time evolution of the reduced density matrix of the system, determined by the Hamiltonian (3). We rely on the standard methods of the “master equation approach,” according to which the effect of the environment is considered in the Born approximation and the environment is assumed to be at each time in its own thermal equilibrium state at temperature $T$. One has [9]

$$\dot{\rho}(t) = -i \int_0^t d\tau [\hat{A}^\dagger \hat{\rho}(t - \tau) - \hat{A} \hat{\rho}(t - \tau)\hat{A}^\dagger] + g(-\tau)[\tilde{\rho}(t - \tau)\tilde{A}^\dagger \tilde{A} - \tilde{A}^\dagger \tilde{A}\tilde{\rho}(t - \tau)\tilde{A}],$$  

(4)

where $\tilde{\rho}(t)$ denotes the time evolution of the reduced density matrix of the system in the interaction picture—e.g. $\tilde{\rho}(t) = U^\dagger(t,0)\rho(t,0)U(t,0)$, where $U(t,0) = T[\exp{-i\int_0^t dt' H_0(t')}]$ and $T$ is the time-ordered product. In the above equation $\tilde{A}$ and $\tilde{A}^\dagger$ stand for $\tilde{A}(t)$ and $\tilde{A}(t)$ (again, the tilde means time evolution in the interaction picture). In Eq. (4) the effect of the environment is included in the function (for simplicity, $k_B = 1$)

$$g(\tau) = \int_0^\infty J(\omega) \left[ \coth \left( \frac{\omega}{2T} \right) \cos(\omega\tau) - i \sin(\omega\tau) \right] d\omega,$$  

(5)

where the spectral density $J(\omega)$ is defined in a standard way in terms of oscillator parameters in Eq. (3) and its typical behavior, for physical environments in the low-frequency regimes, is proportional to $\omega^s$, with $s \geq 1$ [9]; the asymptotic decay of the real part of $g(\tau)$ defines the characteristic memory time $\tau_E$ of the environment.

We solve Eq. (4) for an environment with memory time $\tau_E$ faster than the time scale of the variation of the density matrix $\tau_D$ (estimated self-consistently), so that $\tilde{\rho}(t - \tau) = \tilde{\rho}(t)$ (Markov approximation). In this approximation, Eq. (4) simplifies and assumes a convenient form—for the numerical analysis we wish to perform—in the Schrödinger representation

$$\dot{\rho}(t) = -i[H_0(t),\rho(t)] - \mathcal{L}(\rho),$$  

(6)

with

$$\mathcal{L}(\rho) = \int_0^\infty d\tau [g(\tau) [\hat{A}^\dagger \hat{\rho}(t - \tau) - \hat{A} \hat{\rho}(t - \tau)\hat{A}^\dagger] + g(-\tau)[\tilde{\rho}(t - \tau)\tilde{A}^\dagger \tilde{A} - \tilde{A}^\dagger \tilde{A}\tilde{\rho}(t - \tau)\tilde{A}]],$$  

(7)

where $\hat{A}^\dagger = U^\dagger(t - \tau, t)AU(t - \tau, t)$ and, in the adiabatic approximation, $U(t - \tau, t) = \exp[i\tau H_0(t)]$. As usual, the upper extreme of integration is extended to infinity because the evolution time is much longer than $\tau_E$.

## III. COMPUTER SIMULATIONS AND RESULTS

In order to estimate how the environment affects the performance of the ideal holonomic gates determined by the “dissipation-free” Hamiltonian $H_{0}(t)$, we use the standard notion of fidelity:

$$\mathcal{F} = |\langle \psi_{\text{adh}}(t_{\text{tad}}) | \rho(t_{\text{tad}}) | \psi_{\text{adh}}(t_{\text{tad}}) \rangle|,$$  

(8)

where $|\psi_{\text{adh}}(t_{\text{tad}})\rangle$ is the state in which the initial (pure) state evolves, in the adiabatic time $t_{\text{tad}}$, under the action only of $H_{0}(t)$, while $\rho(t_{\text{tad}})$ is the solution of Eq. (6), computed at the same time and for the same initial (pure) state. In order to avoid dependence on the latter we have taken a suitable average on the initial states. We make a sampling of the initial logical states (combination of the logical states $|+\rangle$ and $|\rangle$) on the Bloch sphere. We add the possibility of an error in the preparation of the initial state with the population of the nonlogical state $|0\rangle$. This can be due to the imprecise control of the lasers. The initial state has the form $|\alpha\rangle = |+\rangle + |\rangle$ with $|\alpha|^2 + |\beta|^2 + |\eta|^2 = 1$ and $|\eta|^2 = 0.1$. In the following, with a slight abuse of notation, we shall denote by the same symbol $\mathcal{F}$ this averaged fidelity.

An essential ingredient of our analysis is the spectral density entering in Eq. (5). For the electronic states in quantum dots the decoherence effects are principally due to phonons. Single-phonon processes are described by super-Ohmic spectral densities with $J(\omega) = k_{3}\omega^{3}e^{-\omega/\omega_{c}}$ [11]. The high-frequency cutoff $\omega_c$ is due to the planar confinement in the quantum dot. The coupling constant $k_{3}$ allows the description of different kinds of phonon-carrier interactions in semiconductor materials including deformation potential, piezoelectric, and spin-orbit [12].

The results of numerical simulations for the fidelity $\mathcal{F}$ of gate 1, as a function of the temperature, are shown in Fig. 1: at low temperature the fidelity is constant and decreases linearly as the temperature increases. To understand how this behavior comes about it is convenient to express the dissipative part of $\mathcal{L}(\rho)$ in Eq. (6) (in the dark-bright state basis) as

$$\mathcal{L}(\rho) = \int_0^\infty d\tau [g(\tau) [\hat{A}^\dagger \hat{\rho}(t - \tau) - \hat{A} \hat{\rho}(t - \tau)\hat{A}^\dagger] + g(-\tau)[\tilde{\rho}(t - \tau)\tilde{A}^\dagger \tilde{A} - \tilde{A}^\dagger \tilde{A}\tilde{\rho}(t - \tau)\tilde{A}],$$  

(7)

where $\hat{A}^\dagger = U^\dagger(t - \tau, t)AU(t - \tau, t)$ and, in the adiabatic approximation, $U(t - \tau, t) = \exp[i\tau H_0(t)]$. As usual, the upper extreme of integration is extended to infinity because the evolution time is much longer than $\tau_E$. 

052304-2
FIDELITY OPTIMIZATION FOR HOLONOMIC QUANTUM...

FIG. 1. Fidelity $F$ for gate 1 as a function of temperature for different $k_3$ [expressed in (meV)$^{-2}$] in the presence of a super-Ohmic environment. The points are the computer simulation results, and the curves have the form $F = 1 - \sum_j \eta_j V_j / t_{ad}^3$ ($\eta_j = 3 \times 10^{-2}$ and $\omega_k = 0.7$). Inset: fidelity $F$ as a function of $k_3$ for two different temperatures ($T_1/\Omega = 1.6 \times 10^{-2}$ and $T_2/\Omega = 5 \times 10^{-3}$). Parameters: $\epsilon = 1$ eV, $\Omega = 25$ meV, $t_{ad} = 7.5$ ps, and $\omega_k = 0.5$ meV.

$$\sum_{kl} \left( \delta_{nl} \sum_r \Gamma_{nrrk}^+ + \delta_{nk} \sum_r \Gamma_{lrrk}^- - \Gamma_{lmmk}^- - \Gamma_{lmmk}^+ \right) \rho_{kl}.$$ Here, $\Gamma_{lmmk}^\pm = \Gamma_{nk} K_{lmmk}$, with $K_{lmmk}$ depending only on laser parameters and

$$\Gamma_{nk}(\omega_{nk}) = J(\omega_{nk}) \left[ \coth \left( \frac{\omega_{nk}}{2T} \right) \right]_{\pm} 1$$

are the transition rates between $n$ and $k$ states due to phonons. In passing, we note that these are indeed the rates that could be guessed by a straightforward application of Fermi’s golden rule to the interaction terms of Eq. (3). Finally, observe that the frequencies $\omega_{nk} = \epsilon_n - \epsilon_k$ represent the energy differences in the dark-bright space—i.e., $\epsilon_n = \epsilon, \epsilon_k$, where $\epsilon_n$ are given by Eq. (2); for $\epsilon \gg \Omega$, $\omega_{nk} = 0, \epsilon, \Omega^2 / \epsilon$.

Equation (9) shows that, with super-Ohmic spectral density, the only relevant transition is the one with $\omega_{nk} = \Omega^2 / \epsilon$ giving the transition rates $\Gamma^\pm(\Omega^2 / \epsilon)$. In fact, for $\omega_{nk} = \epsilon$ the Gaussian cutoff with $\omega_{nk} = \epsilon$ produces negligible rates, and for degenerate dark state ($\omega_{nk} = 0$) the rates vanish. These considerations, together with the explicit form of the rates given by Eq. (9), provide a compelling explanation of the temperature behavior of the fidelity in Fig. 1. Moreover, we have found that the numerical results are fitted by means of the function

$$F = 1 - t_{ad} \sum_{j=1}^{\infty} \eta_j V_j,$$

where $\eta_j$ are two real parameters. This behavior is also manifest by considering the fidelity as a function of the coupling parameter $k_3$ (see inset in Fig. 1). A similar dependence of the fidelity on the transition rate has been found in Ref. [8]. Note that in that case the authors have only a single-transition process (absorption to higher states or emission to lower states) while we need to take into account both absorption and emission processes for the transition to the higher state. This is due to the different master equation solved. In fact, they solve a strictly second-order master equation, while our Eq. (6) is “self-consistent.” The similarity in the results is derived from the small value of the coupling constant $k_3$ but for higher values the two approximations diverge from each other and the numerical results cannot be fitted by such an elementary function.

Equation (10) is particularly important in that it allows one to predict how the fidelity behaves when the parameters of the system are modified. To this end, first of all note that, by keeping the adiabatic parameter constant ($\Omega_{tad} = \alpha = \text{const}$), the rates $\Gamma^\pm$ become a nontrivial function of the adiabatic time. Then, by writing the rates explicitly, one obtains $\Gamma^\pm \propto 1/t_{ad}^\pm (\epsilon^2/\alpha^2)^{1/2} \exp(-[\alpha^2/(t_{ad}^2 \omega_{nk})]^2)$. Thus, it follows from Eq. (8) that the fidelity should have a pronounced minimum as a function of $t_{ad}$. This behavior is confirmed by the computer simulations presented in Fig. 2. By varying $\Omega$ and $t_{ad}$ (e.g., by acting on the lasers) the position of the fidelity minimum in Fig. 2 can be shifted and the effect of super-Ohmic environment can be suppressed. It seems to us that this is an interesting result.

Before discussing the limitations of this result, we would like to comment on the approximations upon which our analysis relies. As we have already anticipated, in Eq. (6) the Markov approximation is appropriate when the memory time $\tau_E$ is small with respect to the time scale of variation of the density matrix $\rho_D$. Equation (5) leads to the estimate $\tau_E = 1/(2 \pi T)$; while Eq. (9), for a super-Ohmic spectral density and $T \ll \Omega^2 / \epsilon$, leads to $\tau_D = (\epsilon / \Omega^2)^3 / k_3$. Note that the conditions of validity of the Markov approximation, $\tau_E < \tau_D$, easily translate into a temperature regime—namely $T \geq T_M = k_3 (\Omega^2 / \epsilon)^3$. With our choice of parameters, we have $1.2 \times 10^{-3} < T_M / \Omega \leq 1.2 \times 10^{-3}$ (depending on $k_3$ value), which, in our simulations, is a very low temperature.

Since the possibility of suppressing the super-Ohmic effects is indeed surprising, one may wonder whether a similar possibility arises for more general environments—e.g., for Ohmic environments. Though Ohmic environments are typical of baths of conduction electrons [9], even for phonon baths, which are typically super-Ohmic, it seems possible...
that the spectral density contains an Ohmic part; this is presumably due to higher-order contributions such as two-phonon processes [13].

Be that as it may, we found it interesting to extend our analysis to an environment with the spectral density $J_\omega = k_1 \omega e^{-\omega_0 \omega} \omega^2$ with $k_1 \ll 1$. As is easily seen, Eq. (9) for Ohmic rates leads to completely different results. This is due to the presence, in the Ohmic case, of transitions between degenerate states which are absent in the case of a super-Ohmic environment. This difference has striking consequences: the transitions between degenerate states give contribution to the rates (9) which are linear in $T$ (while the transitions between nondegenerate states have the same temperature behavior as the super-Ohmic case). This difference is confirmed by the computer simulations in Fig. 3, which shows the fidelity for the super-Ohmic and Ohmic environments and the sum of the two contributions as a function of $T$. This curve clearly shows that the presence of an Ohmic environment changes dramatically the fidelity behavior, whence it follows the impossibility of extending to the Ohmic case the results previously obtained changing the adiabatic time. This conclusion becomes very clear if one compares Fig. 2 with the inset in Fig. 3. It is not possible anymore to optimize the fidelity by changing the parameters.

Before concluding, we briefly mention three points (see [14] for a thorough discussion). First, the computer simula-

FIG. 3. Fidelity $\mathcal{F}$ for gate 1 subject to super-Ohmic (dashed line), Ohmic (dotted line), and both (solid line) spectral densities as a function of $T/\Omega$. Inset: fidelity $\mathcal{F}$ for super-Ohmic plus Ohmic environment as a function of $t_{ad}$ with $\Omega t_{ad} = 280 \left[ k_1 = 4 \times 10^{-4} \text{ and } k_3 = 10^{-1} \text{ (meV)}^{-2} \right]$. Parameters as in Fig. 1.

tion for gate 2 confirms the results found for gate 1. Second, our analysis extends (almost straightforwardly) to the two-qubit gate proposed in Ref. [4]: Figure 4 shows the preliminary results for the two-qubit gate (for a nonrandom initial state) as a function of temperature and coupling constant for super-Ohmic and Ohmic environments; the behavior of the fidelity is analogous to that of the one-qubit case (Fig. 3). Third, for the Ohmic case, a careful study of the (relatively) high-temperature behavior shows the limitations of the Lindblad approximation for the reduced dynamics.

FIG. 4. Fidelity for two-qubit gate subject to super-Ohmic and Ohmic spectral density as a function of $T/\Omega$ for $k_1 = 10^{-4}$ and $k_3 = 10^{-2} \text{ (meV)}^{-2}$. Parameters as in Fig. 1 except $t_{ad} = 0.8 \text{ ns and } \Omega = 0.2 \text{ meV}$.

IV. CONCLUSIONS

To sum up, the upshot of our analysis is twofold. The good news is that it is possible to optimize the fidelity for the kind of environment which is usually considered for electron states in quantum dots—a super-Ohmic environment caused by electron-phonon interactions. The bad news is that such optimization does not go through an Ohmic environment, e.g., produced by the same super-Ohmic phonon bath through two phonon processes. Thus, particular attention should be paid in modeling the environment, since the presence of a weak Ohmic environment dramatically changes the holonomic gate performance. For these reasons it is crucial to dispose of experimental investigations on the nature of the environmental spectral densities in semiconductor quantum dots.


