Pure dephasing due to damped bistable quantum impurities

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Abstract

We study the dynamics of a spin (qubit) coupled to a bistable quantum impurity interacting with a Gaussian bath modeled by a ohmic spin boson model. For white noise the complete dynamics in the four-dimensional Hilbert space is analyzed within a Lindblad formalism. For ohmic damping at finite temperatures we resort to a functional integral approach. We show how different dynamical regimes and crossover of the nonlinear spin boson system are probed by the qubit dynamics.

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1. Introduction

The quantum dynamics of dissipative two state systems has been the subject of intense research in the last thirty years [1,2]. This topic has recently received a great deal of interest in connection with quantum information processing. The extraordinary progress in nanofabrication techniques has in fact paved the way to the observation of coherent effects at the nanoscale [3,4]. Due to the many kinds of low energy excitations typical of the solid state these devices are extremely sensitive to decoherence and dissipation.

Much of the efforts in the analysis of decoherence in quantum logic devices has so far mainly focused on paradigmatic models, like harmonic baths [5,6] or nonlinear spin-baths [7]. In many situations the effect of nonlinear baths, despite of their non-Gaussian nature, can be captured by lowest order correlators of the bath variables, i.e., spectral densities or power spectra [2], this description being exact for harmonic baths. However, a more detailed information on the statistical properties of the environment is needed when the bath has memory on the typical scales of the system dynamics [8,9].

Experiments with solid state qubits [3] have revealed the sensitivity of these nanostructures to decoherence mechanisms which are strictly material, device and protocol dependent. Often these features may be attributed to the interaction with nonlinear and non-Markovian baths. In superconducting qubits background charge fluctuators represent the dominant decoherence source and are presently considered responsible of a variety of effects, ranging from 1/f [10] to random telegraph noise. From recent spectroscopic data, the existence of coherent interaction of a superconducting phase-qubit with a single bistable resonator has been inferred [11].

The analysis of these effects requires the introduction of nonlinear bath models and methodologies which go beyond standard weak coupling approaches [8]. In [9], background charges in superconducting qubits have been modeled as an ensemble of Fano impurities [12] whose incoherent switching produces a bistable fluctuation of
the polarization of the island. A peculiar effect of this environment, related to its nonlinear nature, is that qubit dynamics shows various qualitatively different regimes depending on the switching rate of the impurities, and memory effects are paramount.

Similarly, a qubit coupled to a spin-boson environment may probe different dynamical regimes of the nonlinear bath which can be triggered by changing the temperature and the coupling conditions. In [13], the dynamics of a qubit coupled to a unbiased spin-boson environment in the case of pure dephasing has been studied. Here, we generalize this analysis to a biased spin-boson bath. The environment dynamics is in this regime much richer and manifests itself in different crossovers as a function of the temperature and of the spin parameters. We want to investigate how these effects influence the qubit dynamics. The analysis will evidence regimes of larger sensitivity to the environment dynamics and the appearance of different frequencies in the qubit evolution as a signature of the back-action on the qubit of the coherent quantum dynamics of the spin.

The model we use may describe a Josephson quantum logic device where external voltage gates and magnetic fields are used to tune the effective charging energy $\epsilon$ and Josephson coupling $E_J$ [5,14]. The environment may be a charge close to the device undergoing phonon-assisted tunneling between two positions. We mention finally that the model we study may be relevant for other problems, like two interacting qubits one of them subject to Gaussian noise representing idle qubit not perfectly switched off. As an alternative framework for our work, the qubit may represent a measuring device [15] for the mesoscopic system described by the spin-boson model.

The Hamiltonian is introduced in Section 2 where we also discuss applications to physical problems. In Section 3, we study the coupled qubit–spin dynamics in the presence of white noise using the Lindblad formulation in the Hilbert space of the four states system. We sketch in a phase diagram the various regimes of the dynamics generated by the model. In Section 4, following the influence functional approach of [13], we analyze the finite temperature dynamics of the qubit coupled to an ohmic spin boson system. Finally, we conclude in Section 5.

2. Model

We study the following Hamiltonian ($\hbar = 1$):

$$\mathcal{H} = \mathcal{H}_0 - \frac{1}{2} \hat{X} \tau_z + \mathcal{H}_E,$$  

(1)

$$\mathcal{H}_0 = -\frac{\epsilon_1}{2} \sigma_z - \frac{\nu}{2} \sigma_+ \tau_z - \frac{\epsilon_2}{2} \tau_z - \frac{A}{2} \tau_z.$$  

(2)

It describes two spins ($\sigma$ and $\tau$) coupled to each other by the interaction $\nu$. The spin $\tau$ is also coupled to the environment, described by $\mathcal{H}_E$, via the collective coordinate $\hat{X}$. If the environment is modeled with a set of harmonic oscillators, $\mathcal{H}_E = \sum_i \omega_i \hat{a}_i \hat{a}_i^\dagger$, its effect on the system depends only on the spectral density $G(\omega)$ or equivalently on the power spectrum

$$S(\omega) = \int_0^\infty dt \frac{1}{2} \langle \hat{X}(t) \hat{X}(0) + \hat{X}(0) \hat{X}(t) \rangle e^{i\omega t} = \pi G(\omega) \coth \frac{\beta |\omega|}{2}.$$  

(3)

We consider the standard case when the coupling operator is a collective displacement $\hat{X} = \sum_i \hat{a}_i (a_+ + a_-)$ with ohmic spectral density

$$G(\omega) = \sum_i \gamma_i^2 \delta(\omega - \omega_i) = 2K|\omega|e^{-|\omega|/\omega_c},$$  

(4)

where $\omega_c$ represents the high frequency cut-off of the harmonic modes.

We will first consider an interesting limit of this model, namely high temperature and large cutoff $\omega_c \gg k_b T \gg \Omega_2 = \sqrt{\Delta^2 + \epsilon^2}$. In this limit the spectrum is white, $S(\omega) = 2\Gamma$ where $\Gamma = 2\pi KK_b T$, and it is possible to find solutions which are exact if $T \to \infty$ with $\Gamma$ finite. This limit is very useful in that it allows to sketch in a phase diagram the various regimes of the dynamics generated by the Hamiltonian equation (1), before discussing results for the ohmic spectral density (4).

This model has the symmetry $[\sigma_z, \mathcal{H}] = 0$, therefore $\sigma_z$ is conserved. The dynamics of the coherences $\langle \sigma_{\pm}(t) \rangle$ is conveniently discussed in the basis of the four eigenstates of $\mathcal{H}_0$, which are factorized

$$|a\rangle = |\theta_+ \pm \rangle \otimes |+, \rangle, \quad |c\rangle = |\theta_- \pm \rangle \otimes |-\rangle.$$  

Here, the eigenstates $\sigma_{\pm} |\pm\rangle = \pm |\pm\rangle$ appear. States $|\theta_{\pm} \rangle$, are eigenstates of the conditional (i.e., depending on the state of the spin $\sigma$) Hamiltonians for the spin $\tau$, $\mathcal{H}_\pm = -\frac{\nu}{2} (\epsilon_1 \pm \nu) \tau_z - \frac{A}{2} \tau_z$. The splittings in the conditional Hamiltonians are $\Omega_\pm = \sqrt{(\epsilon_1 \pm \nu)^2 + A^2}$ and the eigenvectors expressed in the basis of $\tau_z$, have the standard form depending on the conditional bias $\cot \theta = (\epsilon_1 \pm \nu)/A$. The spectrum of the full Hamiltonian is formed by two doublets at $E_{\pm \pm} = -\frac{1}{2} (\epsilon_1 \pm \Omega_\pm)$ and $E_{\pm \mp} = \frac{1}{2} (\epsilon_1 \mp \Omega_\pm)$.

Although levels may cross as a function of the parameters, this is not important for our problem since the environment only induces transitions within each doublet. However, the coherences of the open four state system may involve states with almost equal splittings. For instance the conditional frequency $\delta = \Omega_+ - \Omega_-$ may become small when $\nu$ is small or even when $\theta = \frac{1}{2} (\theta_+ + \theta_-) \approx \pi/2$. These are situations where the calculation of decoherence rates for the systems are critical even for very weak damping, due to the effect of non-secular terms.

As a result this model is very rich and may describe many different physical problems. The phase diagram is parametrized by $\Gamma$ and by another pair of variables, for instance $(\chi, \theta)$, where $\chi = \frac{2\nu}{\chi_0} - \theta$ and $\chi = \chi - \chi_+$ (see Fig. 1(a)), or $(\delta, \Omega)$, where $\Omega = (\Omega_+ + \Omega_-)/2$. In this work, we focus on the dynamics of the spin $\sigma$ to see how it is...
affected by the presence of the spin $s$, whose dynamics is influenced by the ohmic bath. Physically our results may describe a qubit ($\sigma$) which is dephased by a nearby bistable impurity ($r$) undergoing damped coherent motion. Alternatively $\sigma$ may be seen as a qubit measuring the dynamics of the spin $s$, and non-Markovian, therefore, it awaits techniques based on low-order expansions in the coupling $v$. Such an approach should eventually factorized and in equilibrium is quite strong if the dynamics of the qubit $s$ becomes slow and does not average.

If the correlation function in Eq. (6) has a range $\tau_C$, then the validity of this standard approach is limited to couplings $v \ll 1/\tau_C$ [8]. In particular, if the spin $s$ has a slow dynamics (for instance for large damping $\Gamma$) we have to resort to other methods.

One possibility is mimicking the spin $r$ plus ohmic bath with a suitable set of harmonic oscillators having the same power spectrum $S_r(\omega)$. In this case the decay is non-exponential for times $t \ll \tau_C$ and eventually turns to the standard exponential decay for $t \gg \tau_C$ [16]. However, previous studies on a similar model have shown that this Gaussian approximation may give good results even for $v\tau_C > 1$ but for shorter and shorter times, as long as $\tau_C$ increases [9]. Moreover, the usual assumption of environment initially factorized and in equilibrium is quite strong if the dynamics of the qubit $s$ becomes slow and does not average.

From a different perspective the failure of the Gaussian approximation can be understood by viewing the qubit $\sigma$ as a measuring device [15] for the mesoscopic system described by the spin-boson model involving $r$. A rather rough measurement protocol (short times, averaging of results) makes the dynamics of $\sigma$ essentially sensitive only to $S_r(\omega)$, whereas if the SB has a slow dynamics the spin $\sigma$ is able to detect also details of the dynamics of $s$ which go beyond $S_r(\omega)$, and have to be described with more careful methods, which we present in the following sections. From now on we treat the spin $s$ on the same footing as the qubit $\sigma$ coupled to an environment consisting of a set of harmonic oscillators.

### 3. White noise

For high temperature and large cutoff of the ohmic bath, $\omega_x \gg k_B T \gg \Omega$ in Eqs. (3) and (4), the power spectrum is white, $S(\omega) = 2\Gamma$. In this limit, we can write a master equation of the Lindblad form for the RDM of the two spins

$$\partial_t \rho(t) = -i[H_0, \rho] - \frac{\Gamma}{2} (\rho - \rho \tau_{\varepsilon}) \leftrightarrow \partial_t \tilde{\rho}(t) = \mathcal{L} \tilde{\rho}(t). \quad (7)$$
where we have introduced the superoperator $\mathcal{L}$, and the column vector $\vec{\rho}$ containing all the elements of the RDM. The 16 eigenvalues of $\mathcal{L}$ express the multi-exponential dynamics of $\rho$. It is convenient to work in the basis of the eigenstates of $\mathcal{H}$, where $\mathcal{L}$ is represented by a $16 \times 16$ matrix, with a rather simple block diagonal structure. Indeed, due essentially to the symmetry $\sigma_z$, the matrix $\mathcal{L}$ presents four $4 \times 4$ diagonal blocks.

### 3.1. Conditional dynamics of the spin $\tau$

Two blocks of $\mathcal{L}$ allow to calculate averages of the eight basis operators of the Liouville space of the two spins $\frac{1}{2} \langle \sigma_i \sigma_k \rangle$ for $i = 0$, $z$ and $k = 0, x, y, z$. Here, we denoted with $\sigma_0$ and $\tau_0$ the identities acting on the Hilbert space of each spin. Therefore, these blocks describe the dynamics of $\sigma_z$, which is fixed to $\pm 1$ and the corresponding conditional dynamics of $\tau$. It is interesting to sketch briefly the main results in this case. Two eigenvalues, associated to the constant quantities $\mathrm{Tr} \rho$ and $\langle \sigma_z \rangle$ vanish. Then we have two groups of three eigenvalues. We show in Fig. 1(b) the typical behavior of the eigenvalues. The phase diagram depends on $x = \Gamma$ and on the conditional bias $y_+ = |\cot \theta_0|$. Here, we discuss the conditional dynamics of $\tau$ when the qubit $\sigma$ is in the state $|+\rangle$. The eigenvalues correspond to the standard ohmic spin-boson problem in the Markov regime for a spin with bias $|\epsilon_2 + e|$. For $y_+ < 1/\sqrt{8}$, we can identify two crossover points, which we call for later use $x_{\mathrm{II}}$ and $x_{\mathrm{IV}}$. For weak damping $x < x_{\mathrm{II}}$, the dynamics is coherent, being given by one real and two complex conjugate eigenvalues. The same situation occurs in principle also in the strong damping regime, $x > x_{\mathrm{IV}}$, but in practice the real eigenvalue dominates the dynamics. Therefore, the spin $\tau$ relaxes with a single rate approximately given by $\Gamma^{-1} \approx \Delta^2 / \Gamma$. This slowing down of the dynamics for increasing damping is a manifestation of the Zeno effect. An intermediate regime also may occur if $x_{\mathrm{II}} < x < x_{\mathrm{IV}}$, where the three eigenvalues are real, and the spin $\tau$ experiences multi-exponential relaxation dynamics. This region disappears when $y_+ > 1/\sqrt{8}$, and in this case we have a single crossover at $x_{\mathrm{II}} = x_{\mathrm{IV}}$ between a weak damping coherent regime and a strong damping Zeno regime.

It is clear that in principle the dynamics of the qubit $\sigma$ may explore a rich environment, exhibiting all the combinations of regimes of the two conditional dynamics of the spin $\tau$.

### 3.2. Coherences of the qubit $\sigma$

We denote with $\mathcal{L}_\phi$ the third block of $\mathcal{L}$, acting on the column vector $\vec{\rho}_\phi$ whose components are $\rho_{ac}$, $\rho_{bd}$, $\rho_{ad}$, $\rho_{bc}$. The fourth block acts separately on the vector $\vec{\rho}_\phi$, formed by the complex conjugate entries of the RDM and gives no further information. The solution of the Lindblad equation $\dot{\vec{\rho}}_\phi = \mathcal{L}_\phi \vec{\rho}_\phi$ gives averages of the remaining eight basis operators of the Liouville space of the two spins $\frac{1}{2} \langle \sigma_i \sigma_k \rangle$ for $i = x, y$ and $k = 0, x, y, z$. Therefore, these blocks describe both $\langle \sigma_- \rangle$ and correlations, including entanglement.

We can express $\mathcal{L}_\phi = (i\Omega - \frac{\chi}{2} I) - \frac{\chi}{2} \mathcal{M}$. The explicit form of $\mathcal{M}$ is easily analyzed after a transformation $\mathcal{M} \rightarrow \mathcal{F} \mathcal{M} \mathcal{F}^{-1}$ in the proper four-dimensional space. The transformed matrix allows to preserve the form of the Lindblad equations for the new vector $\hat{\Sigma} = \mathcal{F} \hat{\rho}_\phi$, with components $\hat{\Sigma} \equiv (\Sigma_1, \Sigma_0, \Sigma_-, \Sigma_+) = (\rho_{ac} - \rho_{bd}, \rho_{bd} + \rho_{ac}, \rho_{ad} - \rho_{bc}, \rho_{ad} + \rho_{bc})$. We find

$$
\mathcal{F} \mathcal{M} \mathcal{F}^{-1} = 
\begin{pmatrix}
-\cos \tilde{\theta} & -i \tilde{\theta} & 0 & \sin \tilde{\theta} \\
-i \tilde{\theta} & -\cos(2\chi) & \sin(2\chi) & 0 \\
0 & \sin(2\chi) & \cos(2\chi) & -i \frac{\Omega}{\Delta} \\
\sin \tilde{\theta} & 0 & -i \frac{\Omega}{\Delta} & \cos \tilde{\theta}
\end{pmatrix}
$$

(8)

### 3.2.1. Weak and intermediate damping

The form (8) suggests the approximation of neglecting all the sine terms, which is valid in many regions of the parameter space, where the modula of the differences of the eigenvalues resulting from the approximation are larger than the off diagonal terms neglected. Then the eigenvalues can be classified in pairs, each of them having the form $\tilde{\lambda}_{1/2} = -\frac{\Gamma}{2} (1 + y_0) \mp \frac{i}{2} \delta_{\mathrm{eff}}$; $\tilde{\lambda}_{3/4} = -\frac{\Gamma}{2} (1 + y_0) \mp i \tilde{\Omega}_{\mathrm{eff}}$.

$$
\tilde{\lambda}_{1/2} = -\frac{\Gamma}{2} (1 + y_0) \mp \frac{i}{2} \delta_{\mathrm{eff}}; \quad \tilde{\lambda}_{3/4} = -\frac{\Gamma}{2} (1 + y_0) \mp i \tilde{\Omega}_{\mathrm{eff}}.
$$

(9)

The important parameter $y_0$ in this approximation reads $y_0 = \cos \theta_+, \cos \theta_+ = (\tilde{\epsilon}_2 - \tilde{\epsilon}_0) / (\tilde{\Omega} - (\delta/2)^2)$. Later we will give more accurate and general expressions. The renormalized frequencies are

$$
\delta_{\mathrm{eff}} = \delta \sqrt{1 - \frac{S \Gamma}{\tilde{\Omega}}}; \quad \tilde{\Omega}_{\mathrm{eff}} = \tilde{\Omega} \sqrt{1 - \frac{(S \Gamma)^2}{\tilde{\Omega}^2}},
$$

where $S = \sin \theta_+ \sin \theta_-$. Therefore, for increasing damping pairs of eigenvalues turn from complex conjugate to two real different eigenvalues. As a function of the damping strength $\Gamma/(2\Delta)$ we can identify two crossover values, $x_1 < x_{\mathrm{IV}}$, corresponding to values of $\Gamma$ roughly proportional to the two scales $\delta < \tilde{\Omega}$. The approximate values are obtained when the arguments of the square roots vanish (see Figs. 2 and 3).

### 3.2.2. The route to strong damping

The analysis of the general case may be guided by the above weak damping approximation. We show illustrative results in Fig. 2, where the essential message is that for large damping eigenvalues belonging to different pairs tend to mix, determining the appearance of two new crossover points. One of them ($x_{\mathrm{II}}$) is defined by the value of $x$ at which the real parts of the two smallest (in absolute value) eigenvalues join. The other one ($x_{\mathrm{IV}}$) refers to the other pair of eigenvalues. It is easy to verify that for $\tilde{\epsilon}_2 = 0$ (Fig. 2(a)) $x_1$ vanishes and $x_{\mathrm{IV}} \rightarrow \infty$, and there is an exact eigenvalue
\[
\lambda_1 = i\epsilon_1 - \Gamma. \quad \text{For } v/\Delta = 1/\sqrt{8}, \text{ we find } \chi_I = \chi_{III} \text{ and the eigenvalues are all real, three of them being equal.} \quad \text{For larger values of the coupling there is a smooth crossover at finite } x \text{ which is mostly seen in the imaginary parts. This regime is interesting if } \tau \text{ is a damped impurity, which may be slower than the coupling, } \Delta < v. \]

When \( \epsilon_2 \neq 0 \) the behavior is classified according to the parameter \( \gamma_0 \). For \( \gamma_0 < 0 \) the eigenvalues with imaginary part \( \Omega_{III} \) dominate (Fig. 2(b)), whereas the other pair dominates if \( \gamma_0 > 0 \) (Fig. 3(a)). At stronger damping \( \gamma_0 \) determines the order of the crossover points, namely \( \chi_{III} < \chi_{IV} \) if \( \gamma_0 < 0 \) and vice-versa. The various regions of the phase diagram for fixed \( (\delta, \Omega) \) can be characterized by a sequence indicating, for increasing \( x \), the presence of a certain number of complex conjugate eigenvalues.

The strong coupling regime, and in particular the eigenvalues with smallest (in absolute value) real part, can be discussed by a suitable rotation of \( \Sigma \). We do not enter into details except quoting the limiting behavior for \( x \to \infty \) of the two interesting eigenvalues, \( \lambda_{x}\ll \approx -\Gamma_x \pm i\Omega_x \), where \( \Gamma_x \approx \Delta^2/(2\Gamma), \Omega_x \approx \Omega \sin \theta \sin \gamma + \delta \cos \theta \cos \gamma \). More accurate expressions valid for \( x > \min(|\chi_{III}, \chi_{IV}|) \), which also estimate very well the crossover points are reported in Section 4.2. Here we point out that for the spin \( \tau_x \) in the strong damping regime we find slowing down of decay for increasing damping, but the eigenvalues involved are now two complex conjugate ones. This result has interesting implications for the dynamics of the coherence \( \langle \sigma_- \rangle \).

### 3.2.3. Time evolution of the coherences

This analysis shows how the phase diagram of the model can be rich, especially for intermediate damping strength. Of course some of these crossover lines are not visible in the dynamics of \( \langle \sigma_- \rangle \) either because the corresponding eigenvalues are not present in the explicit expression of this quantity, or because they are present in terms which are inessential (small amplitude fast decaying) contributions. Therefore, a direct study of the behavior of \( \langle \sigma_- \rangle \) is necessary. We are interested to slowly decaying components of the signal so it is important to focus on the dominant eigenvalues. In our problem they come either in complex conjugate pairs, providing oscillating modulation of \( \langle \sigma_- \rangle \), or as a single isolated real eigenvalue, describing relaxation-like contributions.

Now we roughly sketch the picture for the case \( \gamma_0 < 0 \) and \( v < \Delta/\sqrt{8} \), corresponding to the eigenvalues in Fig. 2. More accurate analytical and numerical results can be easily obtained and generalizations to other cases can proceed along the same lines. Moreover, the same picture for the dynamics holds also for an ohmic environment,
as we discuss later. The regime we analyze has three in principle interesting crossover points, namely \( x_1 \lesssim x_{1\mathrm{II}} \lesssim x_{1\mathrm{III}} \).

For very weak damping, \( x < x_1 \) the superoperator \( \mathcal{L}_\phi \) is approximately diagonal in the basis of the vector \( \vec{\rho}_q \). Using approximate eigenvectors we obtain the expression

\[
\langle \sigma_- \rangle \approx e^{i\theta_1 t} \left\{ \cos \chi \left[ \rho_{\text{ac}}(0) e^{-i\Delta_2 t} + \rho_{\text{bd}}(0) e^{i\Delta_2 t} \right] e^{-iR_{\text{d}}_{12} t} + \cos \chi \left[ \rho_{\text{bd}}(0) e^{-i\Delta_2 t} - \rho_{\text{bc}}(0) e^{i\Delta_2 t} \right] e^{-iR_{\text{d}}_{23} t} \right\}.
\]

Both contributions decay slowly, but the one proportional to \( e^{-i\chi t} > 0 \), the dynamics of \( \langle \sigma_- \rangle \) will show essentially a modulation leading to beats at the scale of the conditional splitting \( \delta_{\text{eff}} \), in practice the result for the undamped system. For \( x_1 < x < x_{1\mathrm{II}} \) we can approximately diagonalize \( \mathcal{L}_\phi \) in the basis \( (\Sigma_1, \Sigma_0, \rho_{\text{ad}}, \rho_{\text{bc}}) \) and obtain

\[
\langle \sigma_- \rangle \approx e^{i\theta_1 t} \left\{ \cos \chi \left[ \rho_{\text{ac}}(0) + \rho_{\text{bd}}(0) \right] e^{-i\Delta_2 t} - \sin \chi \left[ \rho_{\text{bd}}(0) e^{-i\Delta_2 t} - \rho_{\text{bc}}(0) e^{i\Delta_2 t} \right] e^{-iR_{\text{d}}_{12} t} \right\},
\]

which shows that the signal is essentially damped with the dominant rate \( \Delta_2 \). The same qualitative result is obtained for \( x_{1\mathrm{II}} < x < x_{1\mathrm{III}} \), which means that the crossover at \( x_{1\mathrm{II}} \) is immaterial for the qubit \( \sigma \). This crossover roughly corresponds to the crossovers to the Zeno regime in the conditional dynamics of \( \tau \), occurring at \( x_{1\mathrm{II}} \) or \( x_{1\mathrm{IV}} \). Therefore, the qubit \( \sigma \) is not a good detector for this crossover. What happens at strong damping \( x > x_{1\mathrm{III}} \) is instead very counterintuitive. If we consider only slowly decaying terms we obtain

\[
\langle \sigma_- \rangle \approx \frac{1}{2} e^{i\theta_1 t - i\Delta_2 t} \left\{ |F_+(0) + G_-(0)| e^{i\Delta_2 t} \right\}.
\]

Therefore, the spin \( \tau \) modulates the phase of the qubit \( \sigma \) at a frequency \( \Omega_{\text{z}} \), containing information on the energy scales of the spin \( \tau \) and the coupling, but not on damping, despite of the fact that in this regime \( \tau \), after a fast transient, is slowly relaxing along \( \tilde{z} \).

### 4. Ohmic damping and finite temperatures

We now study the dynamics of the qubit \( \sigma \) coupled to a nonlinear environment consisting of a Ohmic Spin-Boson model at finite temperatures. A possible approach is to write a standard quantum master equation in the four-dimensional Hilbert space which directly generalizes Eq. (7) and it is valid in the limit of weak coupling with a short-time correlated environment. Here we use a different approach, namely we consider an exact formal expression for the qubit coherence \( \langle \sigma_- (t) \rangle \) only, expressed as a real-time path-integral, which was derived in [13]. This is a systematic technique which allows to deal only with the time scales relevant to the dynamics of the qubit. Results can be conveniently interpreted in the picture sketched by solving the Lindblad equation (7).

We specify our analysis to a factorized initial density matrix \( \rho(0) = \rho_{\text{ad}}(0) \otimes \rho_{\text{c}}(0) \), where the spin \( \tau \) is initialized in the mixed state \( \rho_{\text{c}}(0) = \frac{1}{2} \hat{I} + \frac{1}{2} \delta \rho(0) \tau_z \).

The qubit coherence is related to the following correlation function involving the spin-boson variables [13]

\[
\frac{\langle \sigma_- (t) \rangle}{\langle \rho_- \rangle} = e^{i\theta_1 t} \mathbb{T} \mathcal{S}_B \left( e^{-iX_{\mathcal{S}B} t} \rho_{\text{c}}(0) \otimes w_{\beta} e^{iX_{\mathcal{S}B} t} \right) e^{i\theta_1 t} = e^{i\theta_1 t} C_{\tau\tau}(t),
\]

where \( w_{\beta} \) denotes the thermal equilibrium state of the bosonic environment and \( \mathcal{S}_B = \mathcal{S}_B \pm \frac{\pi}{2} \tau_z \). In [13] it has been shown that the Laplace transform of the correlator \( C_{\tau\tau}(t) \) reads:

\[
\widehat{C}_{\tau\tau}(\lambda) = \frac{1}{D(\lambda)} \left\{ |K_1(\lambda) - i e^\delta p(0)| \right\},
\]

\[
D(\lambda) = \lambda^2 + \lambda^2 + i K_1(\lambda) + i K_2(\lambda),
\]

where \( K_1(\lambda) \) and \( K_2(\lambda) \) are the Laplace transforms of the kernels given in the Appendix, Eq. (37). The lowest order contributions do not depend on the coupling \( v \) and coincide with the lowest order kernels entering the dynamics of \( \tau \) in the uncoupled case \( (v = 0) \) [2]:

\[
\mathcal{X}_1(\tau - t') = A^2 \cos(\pi K) G_1(\tau - t') \cos[\epsilon_2(\tau - t')],
\]

\[
\mathcal{X}_2(\tau - t') = A^2 \sin(\pi K) G_1(\tau - t') \sin[\epsilon_2(\tau - t')],
\]

where \( G_1(t) = \exp\{-Q(t)\} \) and \( Q(t) \) is the second integral of the bath correlation function

\[
Q(t) = Q(t) + iQ'(t)
\]

\[
= \int_0^\infty \frac{d\omega}{\omega} \frac{G(\omega)}{\omega^2} \left\{ \coth \left( \frac{\omega \beta}{2} \right) (1 - \cos(\omega t)) + i \sin(\omega t) \right\}.
\]

For weak ohmic damping, \( K \ll 1 \) and temperatures \( \Omega_{\text{SB}} \ll k_B T \ll \omega_{\text{c}} \), the characteristic fluctuations of the harmonic oscillators in \( \mathcal{S}_B \) have no memory and \( Q(t) \) assumes the Markov form

\[
Q(t) = 2K \left\{ \frac{\pi |t|}{\beta} + \ln \left( \frac{\beta \omega_{\text{c}}}{2 \pi} \right) \right\} + i\pi K \text{sgn}(t).
\]

The white noise limit discussed in the previous section corresponds to the strict Ohmic regime where \( G(\omega) = 2K\alpha \) and \( S(\omega) \approx 2I \) for frequencies \( \omega \ll 2\pi/\beta \). Modes in this frequency range are responsible for the dynamical term in (18). The logarithmic term is due to oscillators with frequencies \( 2\pi/\beta \ll \omega \ll \omega_{\text{c}} \), which only renormalize the tunneling amplitude \( A_\tau = A_\tau(2\pi k_B T/\hbar)^K \). The imaginary part is typical of Ohmic damping in the scaling limit \( \omega_{\text{c}} t \gg 1 \), where \( Q'(t) = 2K \text{arctan}(\omega_{\text{c}} t) - \pi K \text{sgn}(t) \).

For the Markovian form (18) all contributions to \( K_1(\lambda) \) and \( K_2(\lambda) \) of order higher than \( \lambda^2 \) cancel out exactly, thus the kernels reduce to the lowest order terms Eqs. (16) and (17) which read:

\footnote{In the eigenstate basis of \( \mathcal{S}_B \) we have \( \rho_{\text{c}}(0) \delta \rho(0) = \cos \theta(\rho_{\text{ad}}(0) - \rho_{\text{bc}}(0)) - \sin \theta(\rho_{\text{ad}}(0) + \rho_{\text{bc}}(0)). \)
\[ \mathcal{H}_1(\lambda) = \frac{\lambda + \Gamma}{\epsilon_2^2 + (\lambda + \Gamma)^2}, \tag{19} \]
\[ \mathcal{H}_2(\lambda) = -\pi K A^2 \frac{\epsilon_2}{\epsilon_2^2 + (\lambda + \Gamma)^2}. \tag{20} \]

The white noise limit corresponds to \( T \rightarrow \infty \) keeping \( \Gamma \) finite, thus, the contribution coming from the kernel \( \mathcal{H}_2(\lambda) \propto K \) does not enter the dynamics in the Lindblad formulation. The correlator \( C_{++}(\lambda) \) is readily found as:
\[ C_{++}(\lambda) = \frac{\left[(\lambda + \Gamma)^2 + \epsilon_2^2\right] - \text{Im} \delta \rho(0)}{D(\lambda)} + \frac{\lambda^2}{\epsilon_2^2 + (\lambda + \Gamma)^2}, \tag{21} \]
\[ D(\lambda) = \left[(\lambda + \Gamma)^2 + \epsilon_2^2\right] - \lambda^2/\epsilon_2^2 + \frac{\lambda^2}{\epsilon_2^2 + (\lambda + \Gamma)^2} - i\pi K c_2 A^2. \tag{22} \]

Solving the 4th degree pole equation corresponds to solve the Lindblad problem for the elements \( ac, bd, ad, \) and \( bc \). Differences with this case come from the renormalized tunneling term \( A^2 \) which is included in \( \Omega \) and \( \delta \) by replacing \( \lambda \) with \( A^2 \), and from the \( K \)-dependent imaginary term coming from \( \mathcal{H}_2(\lambda) \). Notice that for orthogonal spins \((\epsilon_2 = 0)\) this term vanishes and the pole equation \( D(\lambda) = 0 \) coincides with the secular equation for the Lindblad matrix \( \mathcal{L}_\phi \) with \( \lambda \rightarrow A^2 \).

In the small damping regime \( \Gamma \ll \Omega \) the poles of \( C_{++}(\lambda) \) have the simple structure
\[ \lambda_{1/2} = \Sigma_1 \mp \sqrt{\Sigma_1^2 - \Pi_1}, \quad \lambda_{3/4} = \Sigma_2 \mp \sqrt{\Sigma_2^2 - \Pi_2}, \tag{23} \]
where
\[ \Sigma_i \approx \Sigma^{(0)}_i + \Sigma^{(2)}_i \Gamma^3, \quad \Pi_i \approx \Pi^{(0)}_i + \Pi^{(2)}_i \Gamma^3 \tag{24} \]
are conveniently expressed in terms of the parameters
\[ \Omega^2 = \sqrt{\frac{\delta^2}{4} - \frac{\Delta^2}{4}} - 4i\pi K c_2 A^2, \quad y = \frac{\epsilon_2^2 - \nu^2 + \Omega^2}{\Omega^2} \tag{25} \]
and read:
\[ \Sigma^{(0)}_1 = \frac{1}{2}(1 - y), \quad \Sigma^{(0)}_2 = \frac{1}{2}(1 + y), \tag{26} \]
\[ \Pi^{(0)}_1 = \frac{1}{2} \left( \frac{\delta^2}{4} + \frac{\Delta^2}{4} - \Omega^2 \right), \quad \Pi^{(0)}_2 = \frac{1}{2} \left( \frac{\delta^2}{4} + \frac{\Delta^2}{4} + \Omega^2 \right), \tag{27} \]
\[ \Sigma^{(2)}_1 = -\Sigma^{(2)}_2 = -\frac{1}{2} y \left( y^2 - \frac{\Delta^2}{\Omega^2} \right) + \frac{\delta^2}{4} \left( \frac{\Delta^2}{\Omega^2} \right)^2 , \tag{28} \]
\[ \Pi^{(2)}_1 = \frac{1}{2} \left( y - \sqrt{y^2 + \frac{\Delta^2}{\Omega^2}} \right) - \Pi^{(0)}_1 y^2, \tag{29} \]
\[ \Pi^{(2)}_2 = \frac{1}{2} \left( y - \sqrt{y^2 + \frac{\Delta^2}{\Omega^2}} \right) - \Pi^{(0)}_2 y^2. \tag{30} \]

We remark that, because of the imaginary term in the damping strength \( K \) we do not get two sets of complex conjugate solutions. Thus in the regime \(|y| \ll 1 \) and \( x > x_1 \) where in the white noise limit we get real solutions, \( \mathcal{H}_1/\lambda_{1/2} \) acquire small extra contributions \( \propto K \), see Fig. 5.

The phase diagram in Fig. 4 illustrates the different dynamical regimes described by the above expressions, and we also compare results for white noise (\( K = 0 \) keeping \( \Gamma \) finite). For finite \(|y| \) we have \(|Re\lambda_{1/2}| \approx |Re\lambda_3| \) and \(|Re\lambda_4| \approx |Re\lambda_5| \). If \( y > 0 \), \(|Re\lambda_{1/2}| > |Re\lambda_5/4| \), the roles of \( \lambda_{1/2} \) and \( \lambda_{3/4} \) being exchanged when \( y < 0 \). With increasing \( K \), differences between the functional integral approach and the Lindblad formulation become pronounced especially for small values of \( \Gamma \), the phase diagram becoming similar to the case \( \delta \rightarrow 0 \). However, we remind that the present calculation is valid for not too low temperatures \( k_B T > \Omega \), i.e., \( \Gamma > 2\pi K \Omega \) (see Fig. 5).

4.1. Regime \( \delta \ll \Omega \)

When \( \delta \ll \Omega \) we may approximate \( \Omega^2 \approx \Omega^2 - \frac{\delta^2}{4} - i\pi KA^2 \frac{\epsilon_2}{\Omega^2} \), and include the imaginary term to linear order in \( K \) into the complex frequencies
\[ F^2 = \Omega^2 + i\pi KA^2 \frac{\epsilon_2}{\Omega^2 - \frac{\delta^2}{4}}, \quad f^2 = \frac{\delta^2}{4} - i\pi KA^2 \frac{\epsilon_2}{\Omega^2 - \frac{\delta^2}{4}}. \tag{31} \]

Fig. 4. Phase diagram of \( \mathcal{H}_1 \) in the plane \( x = \Gamma/2\Omega, y = (\epsilon_2^2 - \nu^2)/4\Omega^2 \) for \( \delta = 0.2\Omega \) (the Markov approximation is valid for \( x > \pi \)). Red corresponds to \(|Re\lambda_1| \ll |Re\lambda_2| \ll |Re\lambda_3| \ll |Re\lambda_4| \), blue to 3.4,1.2, green to 1.3,4,2, cyan to 3,1,4,2 yellow to 1,3,2,4. From left to right we have \( K = 0 \) (white noise), \( K = 0.008, K = 0.03 \). (For interpretation of the references in color in this figure legend, the reader is referred to the web version of this article.)
In this limit, the leading term in \( \lambda_{3/4} \) is \( \Pi_{2}^{(0)} \) which gives
\[
\lambda_{3/4} \approx \Sigma_{2} \mp iF \approx -(1+y) \frac{\Gamma}{2} \mp iF \left[ 1 + \frac{1}{2} \left( \Pi_{2}^{(2)} - \Sigma_{1}^{(0)2} \right) \left( \frac{\Gamma}{2F} \right)^{2} \right].
\]
(32)
The explicit form for \( \lambda_{1/2} \) depends on whether \( \delta > \Gamma \) or \( \delta < \Gamma \) and in the following we give asymptotic expansions in these regimes.

For very small damping, \( \Gamma \ll \delta \ll \Omega \) the leading term inside the square root in Eq. (23) is \( \Pi_{1}^{(0)} \) and we may approximate
\[
\lambda_{1/2} \approx \Sigma_{1} \mp iF \approx -(1-y) \frac{\Gamma}{2} \mp iF \left[ 1 + \frac{1}{2} \left( \Pi_{2}^{(2)} - \Sigma_{1}^{(0)2} \right) \left( \frac{\Gamma}{2F} \right)^{2} \right].
\]
(33)
The linear terms in \( \Gamma \) in Eqs. (32) and (33) corresponds, respectively, to the white noise forms for weak damping given in Eq. (9).

In the special case \( y = 0 \) (i.e., \( v = \epsilon_{2} \)) all the poles give decay on the same scale \( \Re \lambda_{k} \approx - \frac{\Gamma}{2} \) and \( \Pi_{2}^{(2)} - \Sigma_{1}^{(0)2} \approx -(1+2\frac{\delta \tau}{\Gamma \sqrt{2}}) \), \( \Pi_{1}^{(2)} - \Sigma_{1}^{(0)2} \approx -(1-2\frac{\delta \tau}{\Gamma \sqrt{2}}) \). In the limit \( |y| \rightarrow 1^{-} \) (i.e., in the limit regimes \( \epsilon_{2}, A \ll \epsilon_{1} \) or \( v, A \ll \epsilon_{2} \)) only one set of poles dominates the dynamics.

In the limit \( \delta \ll \Gamma \ll \Omega \) we have two sets of complex conjugate poles where, as a difference with the white noise case, \( \lambda_{1/2} \) have a very small imaginary part \( \mp i\pi K_{0} \frac{\epsilon_{1}}{\epsilon_{1}+\epsilon_{2}} \). For \( y \rightarrow 0^{-} \) the dynamics is dominated by the scales \( \lambda_{3/4} \) which reflect the coherent dynamics of \( \tau \) for \( \epsilon_{2} \rightarrow 0 \) and weak damping. Instead for \( y \rightarrow 0^{+} \) the system incoherently decays with the real rate \( \Re \lambda_{1} \). This is the regime of very weak coupling between the qubit \( \sigma \) and the SB environment where the dynamics of \( \sigma \) only depends on equilibrium correlation functions of the spin \( \tau \) and the qubit dephases on a scale given by the golden rule rate
\[
\frac{1}{T_{2}^{*}} = \frac{1}{2} S_{\tau}(\omega = 0) = \frac{\epsilon_{2}^{2}}{2} S_{\tau}(\omega = 0),
\]
(34)
where \( S_{\tau}(\omega) \) is the equilibrium correlation function of \( \tau \) in the spin-boson model. In the Markov limit the so-called non-interacting-blip-approximation (NIBA) is exact \([1,2]\) and we simply get
\[
\tilde{S}_{\tau}(\omega = 0) = 2 \frac{\epsilon_{2}^{2} + \Gamma^{2}}{\lambda_{1}^{2}} \left[ 1 - \langle \tau_{z} \rangle_{\infty} \frac{\Delta^{2} + \epsilon_{2}^{2} + \Gamma^{2}}{\epsilon_{2}^{2} + \Gamma^{2}} \right],
\]
(35)
where \( \langle \tau_{z} \rangle_{\infty} = -\frac{\pi K_{0}}{\Gamma} \) is the equilibrium value of \( \langle \tau_{z} \rangle \) for \( v = 0 \), this term explicitly depends on \( K \) and it cannot be obtained in the white noise limit. The asymptotic limit (34) with (35) is indeed recovered from the pole \( \lambda_{1} \) in Eq. (23) in the damping regime \( \delta \ll \Gamma \ll \Omega \) for \( y > 0 \).

4.2. Strong damping

The behavior of the poles for strong damping \( \Gamma \gg \Omega \) can be obtained by neglecting the imaginary term \( -i\pi K_{0} \epsilon_{1} A_{2}^{2} \) in the pole Eq. (22) and inserting the ansatz
\[
\lambda_{1/2} = -\frac{\Gamma}{2} + \frac{2 \delta^{2}}{T} \mp i \sqrt{1 - \left( \frac{\Gamma_{1}}{\Gamma} \right)^{2}},
\]
\[
\lambda_{3/4} = -\frac{\delta^{2}}{2T} \mp i \sqrt{1 - \left( \frac{\Gamma_{II}}{\Gamma} \right)^{2}},
\]
(36)
which allows to find \( \Gamma_{1} = A \sqrt{-1 + (A/2\epsilon_{2})^{2}} \) and \( \Gamma_{II} = A \sqrt{-1 + (A/2\epsilon_{2})^{2}} \). We remind that the pair of slow eigenvalues \( \lambda_{3/4} \) enter the expression of \( \langle \sigma_{-} \rangle \). The expressions Eq. (36) interpolate the crossover regime \( x > \min (x_{III}, x_{IV}) \) discussed in Section 3.2.2.

5. Conclusions

We have studied the dynamics of a qubit \( \tilde{\sigma} \) coupled to a damped spin, \( \tilde{\tau} \). Damping has been modeled by a set of
harmonic oscillators with ohmic spectral density, therefore the qubit sees a spin-boson system as an environment. This model exhibits a very rich phase diagram and allows to explore several regimes with well controlled techniques.

The dynamics of $\sigma$ reflects the two regimes of conditional dynamics of $x$. The analysis has evidenced regimes of larger sensitivity to the environment dynamics and the appearance of different frequencies in the qubit evolution as a signature of the back-action on the qubit of the coherent quantum dynamics of the spin. The angle $\chi$ modulates the sensitivity of the amplitude of the oscillations of $\langle \sigma_- \rangle$ to the presence of the $x$, whereas phase modulations also depend on damping $\Gamma$. The phase diagram is parametrized by $x = \Gamma/(2d)$, $y_0 = (\epsilon^2 - \epsilon^2)/[\Omega - (\delta/2)^2]$ and the frequencies $\delta \overline{\Omega}$. We summarize in Table 1 the order of the crossover points, where couples of eigenvalues join and the corresponding number of complex conjugate eigenvalues for white noise. Real eigenvalues for white noise acquire small imaginary contributions $iK$ for ohmic damping. From the analysis of the amplitude factors we found that the qubit $\sigma$ is not a good detector of the crossover at $x_{III}$, corresponding to the crossover to the Zeno regime for the conditional dynamics of $\tau$. For strong coupling instead $x > x_{III}$ the spin $\tau$ modulates the phase of the qubit $\sigma$ at a frequency $\Omega_2$, containing information on the energy scales of the spin $\tau$ and the coupling, but not on damping, despite of the fact that in this regime $\tau$, after a fast transient, is slowly relaxing along $\tilde{z}$. The relative order of $[H_{x,i}, x_i]$, $i = 1, 2, 3, 4$ is summarized in Table 2, for different coupling conditions and values of the parameter $y_0$.

Our analysis has evidenced that the spin-boson model, besides describing a variety of physical systems, may represent a useful tool to study model problems where effects like system-environment entanglement, strong frequency shifts and quantum back-action are important.

### Appendix

The irreducible kernels found in [13] read:

$$K_1(t - t') = \mathcal{H}_1(t - t') - \cos(\pi K)$$
$$\times \sum_{n=2}^{\infty} \left(-\frac{A^2}{2}\right)^n \int_t^{t'} dt_{2n-1} \cdots \int_t^{t'} dt_2 \sum_{i_j} \tilde{G}_{nn} B_n D_n,$$

(37)

$$K_2(t - t') = \mathcal{H}_2(t - t') + \sin(\pi K)$$
$$\times \sum_{n=2}^{\infty} \left(-\frac{A^2}{2}\right)^n \int_t^{t'} dt_{2n-1} \cdots \int_t^{t'} dt_2 \sum_{i_j} \tilde{G}_{nn} B_n D_n.$$

(38)

The irreducible influence functional are given by

$$\tilde{G}_n = G_n - \sum_{j=1}^{n} (-1)^j \sum_{m_1, m_2} G_{m_1} \cdots G_{m_j} \delta_{n_1+m+n_2+m_2+\cdots+m_j},$$

where the inner sum is over all positive integers $m_j$ and

$$G_m = \exp \left[ -\sum_{j=1}^{m} O_{2j+1} \sum_{i=1}^{n} \xi_j \xi_{i,j} A_{i,j,k} \right].$$

The bias factors read $B_m = e^{-i\delta_2 \sum_{j=1}^{n} \xi_j \xi_{i,j}}$, $D_m = \prod_{j=2}^{m} \cos(\pi K \xi_j + \epsilon s_{j-1})$.

### References

Fano impurities are localized levels occupied by electrons which may tunnel to a band of delocalized states, see: G.D. Mahan, Many-particle Physics, third ed., Kluwer, Dordrecht, 2000.


