Anomalous suppression of the shot noise in a nanoelectromechanical system

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In this paper we report a relaxation-induced suppression of the noise for a single level quantum dot coupled to an oscillator with incoherent dynamics in the sequential tunneling regime. It is shown that relaxation induces qualitative changes in the transport properties of the dot, depending on the strength of the electron-phonon coupling and on the applied voltage. In particular, critical thresholds in voltage and relaxation are found such that a suppression below 1/2 of the Fano factor is possible. Additionally, the current is either enhanced or suppressed by increasing relaxation, depending on bias being greater or smaller than the above threshold. These results exist for any strength of the electron-phonon coupling and are confirmed by a four states toy model.

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I. INTRODUCTION

In the last years, nanoelectromechanical systems (NEMS) have been a hot research topic both from the theoretical and the experimental point of view. Combining electronic and mechanical degrees of freedom, NEMS have potentially important applications as fast and ultrasensitive detectors, as conducting phonon cavities, molecules and suspended carbon nanotubes.10,11 At finite bias electrons tend to drive phonons out of equilibrium; signatures of nonequilibrium phonon distribution were observed in a suspended carbon nanotube.10

On the theoretical side, NEMS are often described with simple phenomenological models involving a single electron device coupled to an harmonic oscillator.12-25,27,28 Even within these simple models, many peculiar features such as negative differential conductance,12-14 shuttling instability15,16 and strong mechanical feedback23 have been predicted in the case of an underdamped oscillator. It is then a physically relevant question whether the vibrational energy is reduced by relaxation processes induced by coupling to an external environment27 or rather because tunneling itself.18-20 Up to now, theoretical works have focused mostly on the case of negligible relaxation, taking the opposite case of strong relaxation as a reference term. Significant differences between these two cases have been found both for weak and strong electron-phonon (e-ph) coupling.20,21

Many recent theoretical works have focused on the study of current noise on NEMS. In particular, the Fano factor F, which is the ratio between the zero frequency component of the noise and the average current, has proven to be very sensitive to the e-ph interaction and to the details of the phonon distribution.20,21 A giant enhancement of Fano factor \((F \sim 10^2-10^3)\) has been predicted for strong coupling and negligible phonon relaxation.21 In the opposite limit of strong relaxation, i.e., when the phonons are thermally distributed, a NEMS behaves essentially as a single electron transistor (SET).17

Shot noise in SET has been extensively studied and \(F\) was usually found to be larger or equal to 1/2. However, Fano factors below this limit in the single electron tunneling regime have been predicted in more complicated systems. For instance, coupling to internal degrees of freedom33 can induce \(F\) slightly below 1/2 (\(F \sim 0.45\)). A strong suppression of the noise has been predicted for the quantum shuttle.24,25 In this case, very low values of the Fano factors stem from a highly ordered charge transfer mechanism given by strong correlations between charge and mechanical motion. Experimentally, Fano factors below 1/2 were recently reported for a single level quantum dot and this suppression was associated with the finite bandwidth of the detector.

In this work we discuss how intermediate phonon relaxation influence the transport properties of a SET coupled to a mechanical oscillator. We focus on the sequential tunneling regime and we adopt a rate equation to describe the dynamics of the system. This approach is justified when the characteristic frequency of the oscillator is much bigger than the tunneling rate12,13,20,27 which is the typical experimental situation. We find that finite relaxation rate affects the dynamics in a highly nontrivial way. Both current and noise can be either enhanced or suppressed by relaxation, depending on the e-ph coupling and on the considered voltage range. In particular, for voltages higher than a certain critical value, the Fano factor can be even suppressed below 1/2. This suppression is observed in a completely incoherent regime as a consequence of the interplay between vibration assisted tunneling and direct relaxation of different vibrational states.

The paper is organized as follows. The model Hamiltonian is defined in Sec. II, while in Sec. III we introduce the rate equation and the formal expressions for the current and the noise. In Sec. IV numerical results for the current and Fano factor are presented: in particular, the suppression of the Fano factor is discussed in detail for a wide range of parameter. Finally, analytic expressions for the current and the Fano factor are derived within a toy model employing few phononic states.

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II. MODEL

In several experimental realizations, either using lithographically defined quantum dots\textsuperscript{,4,7} molecules\textsuperscript{8,9} or nanotubes\textsuperscript{10,11} electron transport is dominated by single electron tunneling.\textsuperscript{29} In this case, the system is essentially a SET coupled to an harmonic oscillator. In the low voltage strong Coulomb blockaded regime, double occupancy of the SET is prevented and this can be described as a single electronic level. For sake of simplicity in the following we will consider a spinless model, however this can be readily generalized to take into account spin (see Appendix B). The Hamiltonian of the system then reads $H = H_n + H_b + H_{n,b}$ where\textsuperscript{30,21} ($\hbar = 1$)

$$H_n = \varepsilon n,$$  

(1)

$$H_b = \omega_0 (b^{\dagger}b + 1/2),$$  

(2)

$$H_{n,b} = \lambda \omega_0 (b^{\dagger} + b)n.$$  

(3)

The operator $n = d^{\dagger}d$ represents the occupation number of the single level, whose energy $\varepsilon = \varepsilon_b(V_j)$ can be tuned with the aid of an external gate voltage $V_j$. Vibrational excitations are created by $b^{\dagger}$ and their ground state is defined as the zero-phonon state when $n = 0$. The frequency of the oscillator $\omega_0$ can range from the tens of MHz of a nanometrical cantilever\textsuperscript{44} to a dozen of THz in the case of molecular devices or suspended carbon nanotubes.\textsuperscript{9,10} The dimensionless parameter $\lambda$ in the coupling term $H_{n,b}$ represents the strength of the e-ph interaction. For example, $\lambda = 1$ was reported for the C\textsubscript{60} devices\textsuperscript{8} and for suspended carbon nanotubes,\textsuperscript{11} while values of $\lambda$ between 0.4 and 3 have been found in different C\textsubscript{140} samples.\textsuperscript{9}

The SET is coupled to external leads by a tunneling Hamiltonian

$$H_l = \sum_{k,n=1,2} t_d(c_{k,n,d}^{\dagger} + d^{\dagger}c_{k,n,d}),$$  

(4)

where the operators $c_{k,n}^{\dagger}$ create electrons with momentum $k$ in lead $n = 1, 2$. The leads are described as noninteracting Fermi liquids with

$$H_{\text{leads}} = \sum_{k,n=1,2} \varepsilon_{k,n} c_{k,n}^{\dagger} c_{k,n},$$  

(5)

and their chemical potential can be shifted by a bias voltage $V$. For simplicity, in the following we will assume symmetric voltage drops and symmetric barriers $t_1 = t_2 = t_0$.

Finally, the oscillator is coupled to a dissipative environment that we describe as a set of harmonic oscillators\textsuperscript{35}

$$H_{\text{env}} = \sum_j \omega_j (a_j^{\dagger}a_j + 1/2),$$  

(6)

$$H_{b,\text{env}} = \sum_j \chi_j \omega_j (a_j^{\dagger} + a_j)(b^{\dagger} + b).$$  

(7)

Here $a_j^{\dagger}$ are the creation operators of the bosonic bath modes. The environmental coupling is usefully characterized by its spectral function

$$\mathcal{J}(\omega) = 2\pi \sum_j \omega_j^2 \chi_j^2 \delta(\omega - \omega_j).$$  

(8)

III. RATE EQUATION

The eigenstates of $H_b$ can be written as $|n,l\rangle$, where $n$ denotes the occupation of the single level ($n = 0, 1$) and $l$ the phonon number. The coupling to the leads and to the environment induces an energy broadening of these eigenstates. If this broadening is the smallest energy scale of the problem, a perturbative treatment for $H_l$ and $H_{b,\text{env}}$ is appropriate and a master equation for the reduced density matrix of the system can be derived in the sequential tunneling regime.\textsuperscript{36} At lowest order, the reduced density matrix is diagonal in $n$ but may still be off-diagonal in $l$ because of the e-ph coupling $H_{n,b}$.

In the following we will consider the case where $\omega_0$ is much larger than the bare tunneling rate $\Gamma_v^{(0)} = 2\pi V_0^2$ (with $\nu$ the density of states of the leads). In this “diabatic” regime, which is the typical experimental situation,\textsuperscript{8-10} the elements of the density matrix that are nondiagonal in phonon number become negligible.\textsuperscript{12,13,20,27} Then, the master equation reduces to a rate equation for the occupation probabilities $P_{nl}$ of the state $|n,l\rangle$,

$$\frac{d}{dt} P_{nl} = \sum_{n',l',\alpha} \sum_{\alpha'} \left( \Gamma_{n,l}^{(n',l',\alpha)} - \Gamma_{n',l',\alpha}^{(n,l)} \right) P_{n',l'} + \sum_{l'} \left( P_{nl}^{\text{rel}} \Gamma_{n,l}^{(n',l',\alpha)} - P_{n',l'}^{\text{rel}} \Gamma_{n,l}^{(n',l',\alpha)} \right).$$  

(9)

The coefficients $\Gamma_{n',l',\alpha}^{(n,l)}$ represent the tunneling rates through the $\alpha$th barrier while $\Gamma_{n',l'}^{\text{rel}}$ are the relaxation rates.

In order to evaluate such rates, it is convenient to eliminate the coupling term $H_{n,b}$ from $H_l$ by means of a canonical transformation. Due to the coupling term $H_{b,\text{env}}$, this transformation must include both the operators of the oscillator and those of the environment:\textsuperscript{17}

$$\tilde{O} = e^{\varepsilon \alpha} O e^{-\varepsilon \alpha}; \quad \alpha = \kappa (b^{\dagger} - b) - 2\kappa \sum_j \chi_j (a_j^{\dagger} - a_j),$$

where

$$\kappa = \frac{\lambda}{1 - 4 \sum_j \chi_j^2 \omega_j / \omega_0}.$$  

(10)

The total Hamiltonian is transformed into

$$\tilde{H} = \tilde{H}_n + H_b + H_{\text{leads}} + H_{\text{env}} + \tilde{H}_l + H_{b,\text{env}}$$

where

$$\tilde{H}_l = \sum_{k,n=1,2} t_{\tilde{0}} (c_{k,n,d}^{\dagger} e^{-\varepsilon \alpha} d + d^{\dagger} e^{\varepsilon \alpha} c_{k,n,d})$$

and $\tilde{H}_n = \tilde{\varepsilon} n$ with $\tilde{\varepsilon} = \varepsilon - \lambda \kappa \omega_0$. As the energy of the SET is renormalized by a factor proportional to $\lambda^2$, this represents the relevant parameter for the e-ph interaction.
The transition rates can now be easily calculated using Fermi golden rule, giving rise to tunneling rates proportional to $\tilde{a}_l^2$, and relaxation rates, which depend on $\chi_l^2$.

The relaxation rates represent transitions between vibrational excitations without charge of the electronic state ($\beta_l^\dagger \equiv k_B T$),

$$\Gamma^{rel}_{l\rightarrow (l-1)} = e^{\beta \omega_l} \Gamma^{rel}_{(l-1)\rightarrow l} = \frac{\mathcal{J}(\omega_l)}{1 - e^{-\beta \omega_l}}, \quad (12)$$

where $\mathcal{J}(\omega_l)$ is the spectral density of the phonon bath Eq. (8), evaluated at the frequency of the oscillator. The treating $H_{b,env}$ at second order allows only transitions between neighboring states (i.e., $|l'\!\!-l|\equiv 1$). Transitions with $|l'\!\!-l|\gg 1$ can be included considering different relaxation mechanisms.12,21

The charge transfer rates are induced by $\tilde{H}$, Assuming the electrons in the leads are at equilibrium with their chemical potential, one obtains the following expressions:

$$\Gamma^{0\rightarrow 1}_{l\rightarrow l'} = \Gamma^{0}_{l\rightarrow l'} X_{l'} f_a(\omega_l(l' - l)),$$

$$\Gamma^{1\rightarrow 0}_{l\rightarrow l'} = \Gamma^{0}_{l\rightarrow l'} (1 - f_a(\omega_l(l' - l'))), \quad (13)$$

where $f_a(x) = f(x + \bar{e} - \delta \mu, \alpha)$, $f(x)$ is the Fermi function and $\delta \mu = -(1)^{\alpha+1} eV/2$ is the shift of the chemical potential of the leads induced by the bias voltage. The coefficients $X_{l'}$ are given by

$$X_{l'} = \langle n|e^{-\lambda(b^{\dagger}b)}|n',l'\rangle^2 = e^{-\lambda^2/2} \chi_l^2 \chi_{l'}^2 L_l^{(2)}(\chi_l^2) L_{l'}^{(2)}(\chi_{l'}^2), \quad (15)$$

where $l_\leq = \min\{l, l'\}$, $l_\geq = \max\{l, l'\}$ and $L_l^{(2)}(\chi)$ is a generalized Laguerre polynomial. These terms are called Franck-Condon factors and are well known from molecular spectroscopy.37

The effect of the e-ph interaction on transport is twofold: on the one hand it suppresses the effective tunneling rate (because of the factor $e^{-\lambda^2}$, on the other hand it induces a nontrivial dependence on the phononic indices $l, l'$. Up to moderate e-ph coupling ($\lambda^2 \approx 1$), transitions which conserve or change slightly $l$ have the largest amplitude and those between states with low vibrational number are dominant. Vice versa, the latter are exponentially suppressed for $\lambda^2 \gg 1$, while transitions which change $l$ considerably become favored.

Within the rate equation approach, the current and noise can be evaluated by means of standard techniques.29,31 It is convenient to adopt a matrix formalism and write the rate equation as

$$\dot{\rho} = \mathcal{M} \rho,$$  \quad (16)

where the vector $\rho = \{P_{nl}\}$ represents the time dependent occupation probability distribution. Calling $P^{(0)}$ the stationary solution of Eq. (16), the steady current through the oh barrier is

$$I_{st} = e \langle 1| \mathcal{I}_a| P^{(0)} \rangle, \quad \alpha = 1, 2, \quad (17)$$

where $\langle 1| = (1, 1, \ldots, 1)$, $e$ is the the charge of the electron and $\mathcal{I}_a$ is a matrix representing all the possible transitions through the considered barrier ($\mathcal{I}_a^{\alpha\rightarrow \alpha'} = (-1)^{\alpha+1} (n - n') \Gamma^{\alpha\rightarrow \alpha'}$). Following Ref. 31, the zero frequency component of the noise is given by

$$S_{\alpha, \beta}(0) = -2e^2 \langle 1| (\delta \mathcal{I}_a \mathcal{M}^{-1} \delta \mathcal{I}_\beta + \delta \mathcal{I}_\beta \mathcal{M}^{-1} \delta \mathcal{I}_a) | P^{(0)} \rangle^2 + 2e^2 \delta_{\alpha, \beta} \langle 1| [\mathcal{I}_a, \mathcal{I}_\beta] | P^{(0)} \rangle^2, \quad (18)$$

where $\delta \mathcal{I}_a = \mathcal{I}_a - \langle \mathcal{I}_a \rangle/e$. The last term of Eq. (18) is the Schottky noise produced by tunneling through the junction $\alpha$. Note that the matrix $\mathcal{M}$ is singular, however, all the terms in Eq. (18) are well defined (see Appendix A).

Because of charge conservation, the steady current and the zero frequency current correlators are independent of barrier index, $\langle I_\alpha \rangle = I$, $S_{\alpha, \beta}(0) = S$. In the following we will mostly refer to the Fano factor $F = S/2eI$.

IV. RESULTS

A. Full solution of the rate equation

The dynamics of the system is characterized by two competing time scales: the average time spent by an electron in the dot $\tau_{el}$ and the phonon relaxation time $\tau_{ph}$. If $\tau_{el} \gg \tau_{ph}$, the vibrational excitations tend to relax between each tunneling event to the thermal Bose distribution $P^{eq}_{\beta}(n) = e^{-\beta \omega_n} (1 - e^{-\beta \omega_n})$. In this limit, charge and vibrational degrees of freedom decouple $P_{nl} = P_{n}^{eq} P_{l}$ and the dynamics of the system reduces to an effective two-state sequential tunneling process.37 The analytic expressions for current and noise are well known,30 and for $k_B T \ll eV$ are, respectively, given by

$$I^{eq} = e \frac{\tilde{\Gamma}_1 \tilde{\Gamma}_2}{\tilde{\Gamma}_1 + \tilde{\Gamma}_2}, \quad F^{eq} = \frac{\tilde{\Gamma}_1 + \tilde{\Gamma}_2}{(\tilde{\Gamma}_1 + \tilde{\Gamma}_2)^2}, \quad (19)$$

Here $\tilde{\Gamma}_1 = \Gamma^{0}(\mathcal{S} \Theta f_1(-\omega_0))$ and $\tilde{\Gamma}_2 = \Gamma^{0}(\mathcal{S} \Theta a^\dagger \mathcal{A}^1 \mathcal{A}^2(-\omega_0))$ are the renormalized rates for tunneling in and out of the dot and $a_1$ and $a_2$ are Poissonian weight factors $a_1 = \Theta(l) e^{-\lambda^2} \lambda^2/|l|$.

In this case the smallest possible value of the Fano factor is $F^{eq} = 1/2$.

Vice versa, if $\tau_{el} \ll \tau_{ph}$, the tunneling electrons drive the vibrations out of equilibrium and peculiar features such as negative differential conductance (NDC)12,14,27 and super-Poissonian shot noise34 have been predicted.

In our model, a rough estimate of $\tau_{el}^{-1}$ is given by $\tau_{el}^{-1} = \Gamma^{0}(e^{-\lambda^2})$, i.e., by the effective transparency of the barrier set by the e-ph coupling, while $\tau_{ph}^{-1}$ is determined by environment spectral density $\Gamma_{ph}^{-1} = \mathcal{J}(\omega_0)$. It is useful to define a dimensionless parameter for the relaxation strength

$$w = \mathcal{J}(\omega_0)/\Gamma^{0}.$$ \quad (20)

In terms of $w$, the condition for equilibrated phonons $\tau_{el} \gg \tau_{ph}$ reads $w \gg \exp(-\lambda^2)$. It is then evident that the e-ph coupling defines a characteristic scale for relaxation: the stronger is the coupling, the more sensitive is the system to phonon relaxation.

This is reflected by the stationary phonon distribution $P^{eq}_{l} = P^{eq}_{l} P^{eq}_{l'}$. For increasing relaxation strength $w$, $P^{eq}_{l}$ decreases.
tends monotonically to $P_l^{(eq)}$ but with $\lambda^2$-depending speed (see Fig. 1). For strong e-ph coupling, the phonon distribution is narrow already in the nonrelaxed case\cite{supp} $w=0$ and it reaches equilibrium for values of $w$ which are sensibly smaller than for weak $\lambda^2$.

Since $P_l^{(st)}$ converges monotonically to $P_l^{(eq)}$ for growing $w$, one expects most of the features of the nonequilibrated case to be washed out by increasing relaxation. This is particularly evident in the case of the giant Fano factor observed at low voltages for strong interaction ($\lambda^2 \gg 1$), which is strongly suppressed even by weak relaxation (see Fig. 2). This behavior can be easily understood observing that $F$ depends dramatically on the nonequilibrium distribution of the vibrational excitations induced by tunneling\cite{supp}. In fact, for large $\lambda^2$ transitions between low-lying phonon states are exponentially suppressed [see Eq. (15)]. Therefore, the main contribution to the current comes from high excited vibrational states (states with large $l$) but at low voltages the occupation probability of those states is strongly suppressed\cite{supp, supp2}. These conditions leads to avalanches of tunneling processes which, in turn, are responsible for the huge values of $F$\cite{supp2}. Direct phonon relaxation inhibits this mechanism reducing even further the occupation of states with large $l$ and, consequently, both the current and the Fano factor are strongly suppressed. For very strong relaxation ($w \to \infty$), $F \to 1/2$ as one would expect for equilibrated phonon on resonance ($\bar{\varepsilon}=0$).

Similarly, relaxation has a destructive effect on NDC (not shown) as this is also a consequence of the peculiarity of the nonequilibrium phonon distribution induced by tunneling itself\cite{supp, supp2}.

One could be tempted to conclude that considering explicitly the effects of relaxation simply results in an “interpolating” behavior between the opposite limits of no relaxation and thermally distributed phonons. However, we find that finite relaxation rate can induce unexpected features.

Let us first consider the case of moderate coupling $\lambda^2=3$. In Fig. 3 we plot the Fano factor as a function of voltage for different values of $w$ and for $\bar{\varepsilon}=0$. It appears that $F$ has a nonsystematic dependence on $w$: it can be either enhanced or suppressed by relaxation depending on the considered voltage range. For $eV<6\omega_0$ it is always $F \approx 1/2$. In particular, for $eV<2\omega_0$ it is $F=1/2$ as the tunneling electrons cannot excite vibrations and the system behaves as an ordinary single level. More interestingly, for $eV>6\omega_0$ relaxation can suppress $F$ even below $1/2$. In this voltage range, then, relaxation of the mechanical mode induces a tendency to ordered transfer of electrons through the SET.

It is worthwhile to stress that such a suppression of current fluctuations is not merely an effect of the coupling of the SET to the oscillator, but it is a consequence of a nontrivial interplay between vibration-assisted tunneling and phonon relaxation which induces correlation between different current pulses via emission-absorption of phonons.

This peculiar behavior is observable in a wide range of parameters. Infact, from a numerical analysis, it emerges that
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FIG. 4. Density plot of the Fano factor as a function of bias \( V \) and relaxation \( w \) for different values of \( \lambda \). In all the panels: dark gray \( F < 1/2 \), medium gray \( F = 1/2 \) (indicated by the arrow in the color map) and light gray \( F > 1/2 \). The white line, corresponding to \( F = 1/2 \), represents \( w_r(V) \). The black line in the fourth panel delimits the region where noise is super-Poissonian, \( F > 1 \). Other parameters: \( \bar{e} = 0 \) and \( k_B T = 0.02 \omega_0 \).

\[ F < 1/2 \] can be found for any value of the e-ph coupling. In particular, we observed that it exists a voltage threshold set by the e-ph interaction,

\[ eV_l(\lambda) = 2\omega_0 \text{int}(\lambda^2) \]  

such that, for \( V > V_l(\lambda) \), relaxation larger than a certain threshold value \( w_r(V,\lambda) \) suppresses the Fano factor below \( 1/2 \). This is shown in Fig. 4, which represents a grayscale plot of the Fano factor in the \( (V, w) \) plane, for different values of \( \lambda^2 \). The white contour line corresponds to \( F = 1/2 \) and separates two different regions in the \( (V, w) \) plane: the one to the right of the contour, where \( F < 1/2 \) and the other one where \( F > 1/2 \). In other words, the white line denotes \( w_r \) as a function of \( V \) at given \( \lambda^2 \). The threshold voltage \( V_l(\lambda) \) corresponds to the position of the vertical asymptote of \( w_r(V, \lambda) \).

For \( \lambda^2 < 2 \) the critical voltage coincides with the onset of vibration assisted tunneling \( eV_l(\lambda) = 2\omega_0 \); vice versa for strong e-ph coupling \( (\lambda^2 \gg 1) \), \( V_l(\lambda) \) becomes very large and this is why \( F \) is always higher than \( 1/2 \) in Fig. 2.

The minimal value assumed by the Fano factor \( F_{\text{min}} \) depends itself on the e-ph coupling (see Fig. 5). For weak coupling, \( F_{\text{min}} \) differs only slightly from \( 1/2 \). For stronger coupling \( (\lambda^2 \gg 1) \) it decreases logarithmically and it only reaches the value \( F_{\text{min}} \approx 0.4 \) for considerably strong interactions. Note that each point in Fig. 5 corresponds to different values of voltage and relaxation strength, as the position of \( F_{\text{min}} \) in the \( (V, w) \) plane depends on \( \lambda^2 \). The inset shows the voltage \( V_{\text{min}} \) where the minimum is found.

Finally, let us observe that for \( \lambda^2 > 2 \), the threshold voltage \( V_l(\lambda) \) corresponds to the onset of the transition \( l:0 \rightarrow \text{int}(\lambda^2) \). In this case \( V_l(\lambda) \) is a characteristic voltage also for the current which can be either suppressed or enhanced by relaxation depending on \( V \) being smaller or larger than \( V_l(\lambda) \) (see Fig. 6). Relaxation contributes to populate the low-lying phonon states and then, at low voltages, it inhibits the current as the transitions between those states have exponentially suppressed rates. However, for \( V > V_l(\lambda) \) the transition \( l:0 \rightarrow \text{int}(\lambda^2) \) is allowed and, as it corresponds to the greatest Franck-Condon factor, \( 14 \) it gives a substantial contribution to the current. In this case relaxation has the opposite effect and it sustains the current “feeding” the population of the vibrational ground state. For \( V \sim V_l(\lambda) \), these two mechanisms coexist and, consequently, the current depends only weakly on relaxation (see inset in Fig. 6).

This observation fits nicely what is reported in the literature.\(^{20,21}\) In fact, for \( \lambda^2 < 2 \) the critical voltage is smaller than the energy required to have phonon-assisted tunneling and then the current is enhanced by phonon relaxation at any voltage, consistently to what was observed in Ref. 20. Vice versa for very strong e-ph coupling the enhancement of the

FIG. 5. Main panel: \( F_{\text{min}} \) as a function of \( \lambda^2 \). Each point corresponds to different values of \( w \) and \( V \). Inset: voltage \( V_{\text{min}} \) where the minimum is found as a function of \( \lambda^2 \).

FIG. 6. (Color online) Current as a function of voltage for \( \lambda^2 = 3 \) and for different values of the relaxation strength \( w \). The dotted line \( w = \infty \) corresponds to case of thermally distributed phonons. Other parameters \( \bar{e} = 0 \) and \( k_B T = 0.02 \omega_0 \). Inset: zoom of the plateau around \( eV = 5 \omega_0 \). Current in units \( e \Gamma(0) \).
current due to relaxation can be hardly seen as \( V_f(\lambda) \) shifts to very large voltages.

All the previous results are not qualitatively modified by considering a spin-degenerate electronic level (see Appendix B for further details).

### B. Toy model

To get some insight in these results, we focus on the low voltage region \( eV < 4\omega_0 \) and we consider a toy model with only four accessible states, i.e., \( n=0,1 \) and \( l=0,1 \). In this case analytical expressions for current and noise can be derived. For the sake of simplicity, we report only the solutions on resonance (\( \bar{\epsilon}=0 \) and at zero temperature,

\[
I = e \Gamma^{(0)} \left( \frac{X_{00}}{2} + \bar{\theta}(eV - 2\omega_0) X_{01}(w + 2X_{01} - \Delta) \right) \frac{2}{w + 2X_{01}}
\]

and

\[
F = \frac{1}{2} - \frac{\bar{\theta}(eV - 2\omega_0) X_{01} \Delta [w^2 + w(2X_{01} - \Delta) - X_{01} \Delta]}{(w + 2X_{01})^2 K}
\]

where \( \Delta = X_{00} - X_{11} \) and \( K = [w(X_{11} + \Delta)] + X_{01}(2X_{01} + 2X_{11} + \Delta) \).

From Eq. (22) it is easy to show that the current is an increasing function of \( w \) only for \( \Delta > 0 \) (that is, for \( \lambda^2 < 2 \), see Eq. (15)). Vice versa, for \( \Delta < 0 \) (\( \lambda^2 > 2 \)) the current decreases for increasing relaxation, in agreement with what was previously observed. Moreover Eq. (23) tells that \( \Delta > 0 \) (\( \lambda^2 < 2 \)) is the necessary condition to have \( F < 1/2 \) in the region \( 2\omega_0 < eV < 4\omega_0 \). In fact only in this case, it exist a threshold value for relaxation

\[
2w_f(\lambda) = \Delta - 2X_{01} + \sqrt{\Delta^2 + 4X_{01}^2}
\]

such that for \( w > w_f(\lambda) \) the Fano factor is smaller than \( 1/2 \). For stronger e-ph coupling \( \lambda^2 > 2 \) (\( \Delta < 0 \)), it is always \( F > 1/2 \). This confirms the numerical estimate \( eV_f(\lambda) = 2\omega_0 \) as the threshold voltage for any \( \lambda^2 < 2 \).

Despite the coarseness of the model, Eq. (23) accords qualitatively with the exact numerical solution for \( eV < 4\omega_0 \) (see Fig. 7). The agreement is reasonably good even in the case of weak e-ph coupling, where the phonon distribution is mostly broadened and one expects the four state approximation to be more inaccurate. A better agreement can be obtained considering a six state model with \( n=0,1 \) and \( l=0,1,2 \) but, in this case, the analytic solutions become quite cumbersome and we do not report them here for simplicity. The agreement of the four states model with numerical result suggests that \( F < 1/2 \) rather depends on the interplay between relaxation and vibration-assisted tunneling, than on the possibility to access a high number of vibrational states.

### V. CONCLUSIONS

In conclusion, we have investigated the effects of direct phonon relaxation on the shot noise of a SET coupled to a mechanical oscillator. For increasing relaxation strength, the occupation probability distribution of the states of the system evolves monotonically towards thermal equilibrium. In contrast, we found a nonmonotonous behavior of the Fano factor, which can be suppressed even below 1/2. This relaxation-induced tendency to order of the electronic transfer through the dot is unexpected, since we are dealing with an oscillator with incoherent dynamics coupled to a SET in the sequential tunneling regime. The onset of this behavior is discussed as a function of relaxation, e-ph interactions, and external voltages. We have found that for any value of the e-ph coupling, a critical voltage \( V_f(\lambda) \) exists such that for \( V > V_f(\lambda) \) a suppression of the Fano factor below 1/2 is possible. At low voltages, these results are qualitatively predicted by a four states toy model. In this work we have focused on the case of a symmetric device on resonance, which is the most favorable to analyze the suppression of the Fano factor. However, qualitatively analogous results can be obtained for asymmetric barriers and (or) \( \bar{\epsilon} \neq 0 \).

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### APPENDIX A

In this appendix we comment briefly on the existence of the pseudoinverse \( \mathcal{M}^{-1} \) in Eq. (18).

Let us first note that during numerical calculations, we allow the index \( l \) to take on values between 0 and some large cutoff \( l_{\max} \gg 1 \). In this case \( |P\rangle \) is a vector of length \( 2l_{\max} \) and \( \mathcal{M} \) is a \( 2l_{\max} \times 2l_{\max} \) matrix.

Given the existence of a stationary solution of Eq. (16), \( \mathcal{M}^{\rho(\omega)}=0 \), the matrix \( \mathcal{M} \) is singular. Nevertheless, the quantity \( \langle w | \mathcal{M}^{-1} | x \rangle \) is well defined if: (i) the vector \( | x \rangle \) belongs to the range of \( \mathcal{M} \), (ii) \( \langle w \rangle \) satisfies \( \langle w | \rho(\omega) \rangle = 0 \). The first condition assures that the expression \( \mathcal{M}^{-1} | x \rangle \) is well defined,
being the preimage of $|x\rangle$ under $\mathcal{M}$, i.e., $\mathcal{M}^{-1}|x\rangle=\{y\in\mathbb{R}^{2\max}:\mathcal{M}|y\rangle=|x\rangle\}$.

Since $\mathcal{M}$ has a unique null vector $|P^{(a)}\rangle$, this set can be written as $\mathcal{M}^{-1}|x\rangle=\{y+\alpha|P^{(a)}\rangle, \alpha \in \mathbb{R}\}$, where $|y\rangle$ is an arbitrary solution of the linear system $\mathcal{M}|y\rangle=|x\rangle$. The second condition then assures that the quantity $\langle w|\mathcal{M}^{-1}|x\rangle$ is uniquely defined.

It can be shown that the vector $|x\rangle$ belongs to the range of $\mathcal{M}$ if and only if $\langle 1|x\rangle=0$, being $\langle 1\rangle=\{1,1,1,\ldots,1\}$.

Consequently, all the terms in Eq. (18) are well defined as, by construction, it is $\langle 1|\delta \mathcal{M}_{\alpha}\rangle|P^{(a)}\rangle=0$.

To evaluate Eq. (18) one must solve the linear system $\mathcal{M}|y\rangle=\delta \mathcal{M}_{\alpha}|P^{(a)}\rangle$ by means of standard techniques (e.g., with singular value decomposition) and, finally, to evaluate the scalar product $\langle 1|\delta \mathcal{M}_{\alpha}\rangle|y\rangle$.

### APPENDIX B

In this appendix we consider the case of a spin-degenerate electronic level. Let us consider instead of Eq. (1), the standard Hamiltonian for a spin-degenerate electronic level

$$H_n=\sum_{\sigma=\uparrow,\downarrow} e_{n \sigma} + U n_\uparrow n_\downarrow,$$

where $e_{n \sigma}=d_{n \sigma}^+d_{n \sigma}$ ($\sigma=\uparrow,\downarrow$) and $U$ is the charging energy of the SET. If $U$ is large compared to all other relevant energy scales, double occupancy of the SET is prevented and the electronic level can only be empty or occupied by one electron with either a spin up or down. In this case the rate equation becomes

$$\frac{d}{dt} P_{0\alpha}= \sum_{\sigma,\sigma',\alpha} \left( P_{\sigma\alpha\Gamma_{\sigma'\rightarrow\alpha}}^{\downarrow} - P_{0\alpha\Gamma_{\sigma'\rightarrow\alpha}}^{\uparrow} \right)$$

$$+ \sum_{\sigma'} \left( P_{\sigma\alpha\Gamma_{\sigma'\rightarrow\alpha}}^{\uparrow} - P_{0\alpha\Gamma_{\sigma'\rightarrow\alpha}}^{\uparrow} \right),$$

(B2)

$$\frac{d}{dt} P_{\alpha\sigma}= \sum_{\sigma',\alpha} \left( P_{\sigma\alpha\Gamma_{\sigma'\rightarrow\alpha}}^{\uparrow} - P_{\sigma\alpha\Gamma_{\sigma'\rightarrow\alpha}}^{\downarrow} \right)$$

$$+ \sum_{\sigma'} \left( P_{0\alpha\Gamma_{\sigma'\rightarrow\alpha}}^{\uparrow} - P_{\sigma'\alpha\Gamma_{\sigma'\rightarrow\alpha}}^{\downarrow} \right),$$

(B3)

where $P_{\alpha\sigma}$ is the probability of the state $|\sigma,l\rangle$. In our model, all the transition rates are spin independent, i.e., $\Gamma_{\alpha\rightarrow\alpha'}^{0,\sigma}$

$$=\Gamma_{\alpha\rightarrow\alpha'}^{0,\sigma},$$

and $\Gamma_{\alpha\rightarrow\alpha'}^{\sigma\sigma'}=\Gamma_{\alpha\rightarrow\alpha'}^{\sigma\sigma'}$, where $\Gamma_{\alpha\rightarrow\alpha'}^{0,\sigma}$ and $\Gamma_{\alpha\rightarrow\alpha'}^{1,\sigma}$ are defined in Eqs. (13) and (14). As the total probability for having one electron into the SET is $P_{\uparrow \downarrow}=P_{\uparrow \downarrow}^{\uparrow}+P_{\uparrow \downarrow}^{\downarrow}$, summing over $\sigma$ Eq. (B2) and Eq. (B3), we obtain

$$\frac{d}{dt} P_{0\uparrow}= \sum_{\alpha,\alpha'} \left[ P_{\uparrow\alpha\Gamma_{\alpha'\rightarrow\alpha}}^{\downarrow} - P_{0\alpha\Gamma_{\alpha'\rightarrow\alpha}}^{\uparrow} \right]$$

$$+ \sum_{\alpha'} \left( P_{\alpha\uparrow\Gamma_{\alpha'\rightarrow\alpha}}^{\uparrow} - P_{0\alpha\Gamma_{\alpha'\rightarrow\alpha}}^{\uparrow} \right),$$

(B4)

$$\frac{d}{dt} P_{\alpha\uparrow}= \sum_{\alpha'} \left[ P_{\alpha\uparrow\Gamma_{\alpha'\rightarrow\alpha}}^{\uparrow} - P_{\alpha\uparrow\Gamma_{\alpha'\rightarrow\alpha}}^{\downarrow} \right]$$

$$+ \sum_{\alpha'} \left( P_{\alpha'\uparrow\Gamma_{\alpha'\rightarrow\alpha}}^{\uparrow} - P_{\alpha'\uparrow\Gamma_{\alpha'\rightarrow\alpha}}^{\downarrow} \right),$$

(B5)

which is a set of equations analogous to Eq. (9), except for a factor of 2 in front of all rates which brings one electron into the dot.

Thus, in the case of strong electronic interaction, a spin-degenerate electronic level is formally equivalent to a spinless single level, apart from an effective doubling of the tunneling-in rates. In particular, at low temperature the analytic expressions for current and noise in the limit of strong relaxation, when the system behaves as a single spin-degenerate electronic level, become

$$I_{\pi\pi}=e2\Gamma_{\uparrow\uparrow}/(2\Gamma_{\downarrow\downarrow}+\Gamma_{\uparrow\downarrow})$$

and $F_{\pi\pi}=4\Gamma_{\uparrow\downarrow}/(2\Gamma_{\uparrow\downarrow}+\Gamma_{\downarrow\downarrow})^2$. The lowest value for $F_I$ in this case is not 1/2 but 5/9.

Performing numerical analysis we have found in the case of a spin-degenerate electronic level qualitatively the same behavior observed in the spinless case, i.e., the Fano factor shows a nonmonotonous dependence on the strength of the relaxation rate. In fact, without relaxation the coupling to an harmonic oscillator generally leads to an increase of the current noise with respect to the uncoupled electronic level. However, relaxation of the phononic mode can introduce negative correlation between different current pulses via emission-absorption of phonons and lead to suppression of the Fano factor below 5/9, which is the minimal expected value without e-ph interaction. Moreover, as for the spinless case, we have found that such a suppression is observable only if the source drain voltage is $V>V_{\text{th}}(\lambda)$, see Eq. (21).

For certain choices of the parameters we also registered $F<1/2$, but for the spin-degenerate case such a value does not represent any relevant limit.

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23O. Usmani, Ya. M. Blanter, and Yu. V. Nazarov, cond-mat/0603017 (to be published).
26S. Gustavsson, R. Leturcq, T. Ihn, K. Ensslin, M. Reinwald, and W. Wegscheider, cond-mat/0607192 (to be published).
38The general expression for the weight factors is: $a_i = \exp[-\lambda^2 \coth(\beta_0/2)]\exp(\beta_0/2) I_1(\lambda^2 / \sinh(\beta_0/2))$, where $I_1$ is the modified Bessel function of the first kind.